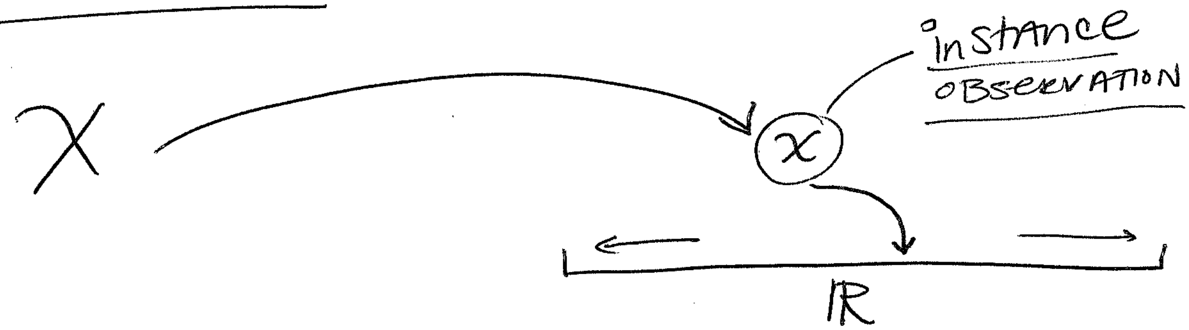
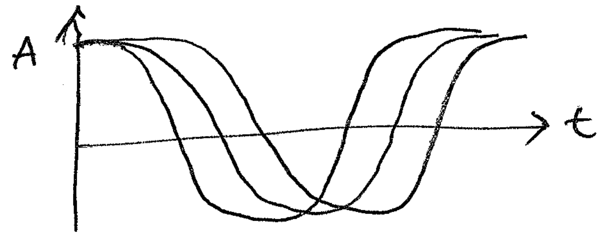
ELEC303 ~ RANDOM PROCESSES

→ finally, we get to "RANDOM signals"

RANDOM VARIABLES



RANDOM PROCESSES



$$X(t) = A\cos(2\pi f_0 t + \theta), \quad \theta \sim \mathcal{U}[0, 2\pi]$$

$$X(t) \longrightarrow x(t \mid \theta)$$

→ instance
→ sample function

given $\theta = \theta^*$,

$$\underbrace{X(t) = A\cos(2\pi f_0 t + \theta^*)}_{\text{deterministic signal}}$$

$$\boxed{\underbrace{X(t_0) = A\cos(2\pi f_0 t_0 + \theta)}_{\text{RANDOM VARIABLE!}}}$$

Random processes
↳ Stochastic process
[counterpart to deterministic system!]

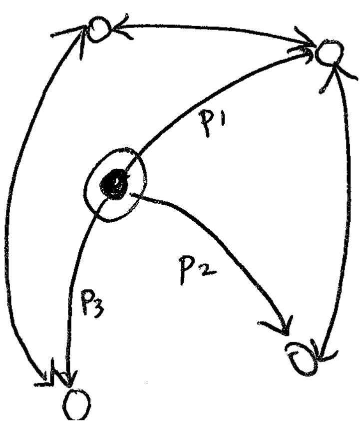
→ instead of describing one way in which a system will evolve over time, the outcome or evolution of time-series is NOT DETERMINED → governed by a prob. distrib.

(1) Discrete time RP → series of RVs (Markov chain)
↓
[next state depends on current state]
↓

$$\text{if @ A, } P(x) = \begin{cases} 0.2 & x='B' \\ 0.3 & x='C' \\ \vdots \end{cases}$$

← modeling of music + note progression

(2) RANDOM FIELD [domain-space]
↳ GRAPH



↳ diffusion
↳ RANDOM walks
→ MOLECULE traveling in LIQUID/GAS
→ SEARCH PATH of A FORAGING ANIM

Now, lets extend previous quantities defined for RVs to random processes

$X(t)$: random process MEAN (1)

$$m_X(t) = \mu_X(t) \equiv E[X(t)] \quad \forall t.$$

At each time instant t_0,

$$m_X(t_0) = E[X(t_0)].$$

↑ RV

PDF of RV $X(t_0) \Rightarrow p_{X(t_0)}(x)$

↓

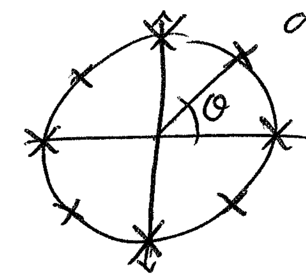
$$E[X(t_0)] = m_X(t_0) = \int_{-\infty}^{\infty} x \, p_{X(t_0)}(x) \, dx$$

EXAMPLE (from before)

$$X(t) = A\cos(2\pi f_0 t + \Theta), \quad \Theta \sim \mathcal{U}[0, 2\pi].$$

$\boxed{PSK}$: Phase-shift keying

mean $X(t) = ?$



discrete $\theta = 0:\frac{\pi}{4}:\frac{7\pi}{4}$

$$E[X(t)] = \frac{A}{2\pi}\int_0^{2\pi} \underbrace{\cos(2\pi f_0 t + \theta)}_{\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)}\, d\theta$$

$$= \frac{A}{2\pi}\int \cos(2\pi f_0 t)\cos(\theta) - \sin(2\pi f_0 t)\sin(\theta)\, d\theta$$

$$= \frac{A}{2\pi}\left(\cancel{\cos(2\pi f_0 t)\sin(\theta)\Big|_0^{2\pi}} + \underbrace{\cancel{\sin(2\pi f_0 t)\cos(\theta)\Big|_0^{2\pi}}}_{= 0}\right)$$

$$= 0.$$

Autocorrelation function

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$= \underline{R_X(t_2, t_1)}$$

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \underbrace{P_{X(t_1),X(t_2)}(x_1, x_2)}\, dx_1\, dx_2$$

joint distribution
over RVs, $\underline{X(t_1), X(t_2)}$.

EX: $X(t) = A\cos(2\pi f_0 t + \theta)$, $\underline{\theta \sim U[0, 2\pi]}$

$R_X(t_1, t_2)$
$= E[A\cos(2\pi f_0 t_1 + \theta)\, A\cos(2\pi f_0 t_2 + \theta)]$

$= A^2 E[\frac{1}{2}(\cos(2\pi f_0(t_1 - t_2)) + \cancel{\cos(2\pi f_0(t_1+t_2)+2\theta)}$

$\Downarrow$

$$\frac{1}{2\pi}\int_0^{2\pi} \cancel{\cos(2\pi f_0(t_1+t_2)+2\theta)} = 0$$

$$= \frac{A^2}{2}\cos(2\pi f_0(t_1 - t_2)).$$

EX:

$X(t) = X \quad , \quad X \sim \mathcal{U}(-1,1)$

$R_X(t_1,t_2) = ?$

$$E[X^2] = \int_{-1}^{1} \frac{x^2}{2}\,dx = \frac{x^3}{6}\Big|_{-1}^{1} = \frac{1}{3}.$$

$$= VAR(uni) + (mean(uni))^2$$

(WSS) WIDE-SENSE STATIONARITY

A process $X(t)$ is WSS if,

(1) $m_X(t) = E[X(t)]$ is independent of t !!

(2) $R_X(t_1,t_2) = R_X(\tau) \, , \quad \tau = t_1 - t_2$

autocorrelation only depends on difference betw. sample pts (not on t_1, t_2)

again for:

$X(t) = A\cos(2\pi f_0 t + \theta)$, $\theta \sim U[0, 2\pi]$.

$\boxed{\text{IS } X(t) \text{ WSS?}}$

$R_X(t_1, t_2) = \frac{A^2}{2} \cos f_0 (t_1 - t_2)$

$m_X(t) = 0$

$\Downarrow$

WSS!

How about $\theta \sim U[0, \pi]$? ~ is $Y(t)$ WSS?

$Y(t) = A\cos(2\pi f_0 t + \theta)$

$m_Y(t) = E[Y(t)] = \int_0^{\pi} Y(t) \cdot \frac{1}{\pi} \, d\theta$.

$= \frac{A}{\pi} \left[\cos(2\pi f_0 t)(\sin\pi - 0) + \sin(2\pi f_0 t)(\underset{-1}{\cos(\pi)} - \underset{-1}{\cos(0)}) \right]$

$= \boxed{-\frac{2A}{\pi} \sin(2\pi f_0 t)} \rightarrow$ not WSS!

Multiple RANDOM Processes

$X(t)$, $Y(t)$ are independent if for all t_1, t_2, RVs $X(t_1)$, $X(t_2)$ Are independent!

[SAme is true for correlation!]

* if independent $\Rightarrow$ uncorrelated
* if uncorrelated $\not\Rightarrow$ independent

COVARIANCE $(X, Y) = E[(X-E[X])(Y-E[Y])]$
if cov=0 $\Rightarrow$ uncorrelated.

$\Rightarrow$ Cross-Correlation $\rightarrow$ auto correlation
$\Rightarrow$ covariance $\rightarrow$ variance

CROSS correlation:

$$R_{XY}(t_1, t_2) = E[X(t_1)\,Y(t_2)]$$

$$R_{XY}(t_1, t_2) = R_{YX}(t_2, t_1).$$

Joint Stationarity — WSS

$X(t), Y(t)$ are jointly WSS if:

(1) both are stationary (WSS)

(2) cross-correlation only depends on $\tau = t_1 - t_2$.