ELEC 303 - RANDOM PROCESSES

- finally, we get to "RANDOM SIGNALS"

RANDOM VARIABLES

\[ X \]

\[ X(t) = A \cos(2\pi f_0 t + \Theta), \quad \Theta \sim U[0, 2\pi] \]

\[ X(t) \]

\[ X(t|\Theta) \]

\[ X(t_0) = A \cos(2\pi f_0 t_0 + \Theta) \]

given \( \Theta = \Theta^* \),

\[ X(t) = A \cos(2\pi f_0 t + \Theta^*) \]

deterministic signal

\[ X(t_0) \]

RANDOM VARIABLE!
Random processes

- Stochastic process
  - Interpret as deterministic system.

→ instead of describing one way in which a system will evolve over time,
the outcome or evolution of time-series is not determined → governed by a prob. distrib.

(1) Discrete-time RP →

Series of RVS (Markov Chain)

[ next state depends on current state ]

if @ A, p(x) = \begin{cases} 0.2 & x = B \\ 0.3 & x = C \end{cases}

modeling of music → note progression

(2) Random Field [domain-space]

- diffusion
- graph

→ RANDOM WALKS
→ MOLECULE traveling in liquid/gas
→ search path of a foraging ant
Now, let's extend previous quantities defined for RVs to random processes.

\[ X(t) : \text{random process} \]

\[ M_x(t) = M_x(t) = E[X(t)] \quad \forall t. \]

At each time instant \( t_0 \),

\[ M_x(t_0) = E[X(t_0)]. \]

PDF of RV \( X(t_0) \) \( \Rightarrow \) \( P_{X(t_0)}(x) \)

\[ E[X(t_0)] = m_x(t_0) = \int_{-\infty}^{\infty} x P_{X(t_0)}(x) \, dx \]
EXAMPLE (from before)

\[ X(t) = A \cos(2\pi f_0 t + \Theta), \quad \Theta \sim U[0, 2\pi], \]

mean \( X(t) = \) ?

\[
E[X(t)] = \frac{A}{2\pi} \int_0^{2\pi} \cos(2\pi f_0 t + \Theta) \, d\Theta
\]

\[
= \frac{A}{2\pi} \int_0^{2\pi} \cos(2\pi f_0 t) \cos(\Theta) - \sin(2\pi f_0 t) \sin(\Theta) \, d\Theta
\]

\[
= \frac{A}{2\pi} \left( \cos(2\pi f_0 t) \sin(\Theta) \bigg|_0^{2\pi} + \sin(2\pi f_0 t) \cos(\Theta) \bigg|_0^{2\pi} \right)
\]

\[ = 0. \]
Autocorrelation function

\[ R_x(t_1, t_2) = E[X(t_1)X(t_2)] \]

\[ = R_x(t_2, t_1) \]

\[ R_x(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 \left\{ \begin{array}{l} \text{joint distribution} \\ \text{over RVS, } X(t_1), X(t_2) \end{array} \right\} dx_1 dx_2 \]

EX: \[ X(t) = A \cos(2\pi ft + \theta), \quad \theta \sim U[0, 2\pi] \]

\[ R_x(t_1, t_2) = E[A \cos(2\pi ft_1 + \theta) A \cos(2\pi ft_2 + \theta)] \]

\[ = A^2 E[\frac{1}{2} \cos(2\pi ft_1 - t_2) + \cos(2\pi ft_1 + t_2 + \theta)] \]

\[ = A^2 \left[ \frac{1}{2} \cos(2\pi ft_1 - t_2) \right], \quad \frac{1}{2\pi} \int_0^{2\pi} \cos(2\pi ft_1 + t_2 + \theta) d\theta = 0 \]
EX:

\[ X(t) = x, \quad X \sim U(-1, 1) \]

\[ R_X(t_1, t_2) = ? \]

\[ E[X^2] = \int_{-1}^{1} \frac{x^2}{2} \, dx = \frac{x^3}{6} \bigg|_{-1}^{1} = \frac{1}{3} \]

= \text{VAR}(\text{uni}) + (\text{mean}(\text{uni}))^2

\hline

WIDE-SENSE STATIONARY

A process \( X(t) \) is WSS if:

1. \( M_X(t) = E[X(t)] \) is independent of \( t \)!!

2. \( R_X(t_1, t_2) = R_X(t), \quad T = t_1 - t_2 \)
   
   \text{Auto-correlation only depends on difference between sample pts (not on } t_1, t_2) \)
again for:
\[ X(t) = A \cos(2\pi f_0 t + \theta), \quad \text{on } U[0, 2\pi]. \]

**IS X(t) WSS?**

\[ R_x(t_1, t_2) = \frac{A^2}{2} \cos f_0 (t_1 - t_2) \]

\[ m_x(t) = 0 \]

**If**

\[ WSS! \]

**How about** \( \text{on } U[0, \pi] \)? \( \sim \) is \( Y(t) \)

\[ Y(t) = A \cos(2\pi f_0 t + \theta) \]

\[ m_y(t) = E[Y(t)] = \int_{0}^{\frac{2\pi}{\pi}} Y(t) \cdot \frac{1}{\pi} \, d\theta. \]

\[ = \frac{A}{\pi} \left( \cos(2\pi f_0 t) (\sin \pi - 0) + \sin(2\pi f_0 t) (\cos \pi - \cos 0) \right) \]

\[ = \frac{-2A}{\pi} \sin(2\pi f_0 t) \rightarrow \text{not WSS!} \]
Multiple Random Processes

$X(t), Y(t)$ are independent
if for all $t_1, t_2$, RVs $X(t_1), X(t_2)$
Are independent!

[Same is true for correlation!]

* if independent $\Rightarrow$ uncorrelated
* if uncorrelated $\not\Rightarrow$ independent

$\text{Covariance} (X, Y) = E[(X-E[X])(Y-E[Y])]$
if cov = 0 $\Rightarrow$ uncorrelated.

$\Rightarrow$ Cross-Correlation $\rightarrow$ Auto-Correlation
$\Rightarrow$ Covariance $\rightarrow$ Variance

Cross-Correlation:

$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$

$R_{XY}(t_1, t_2) = R_{YX}(t_2, t_1)$. 
Joint Stationarity - WSS

$X(t), Y(t)$ are jointly WSS if:

1. Both are stationary (WSS)
2. Cross-correlation only depends on $T = t_1 - t_2$. 