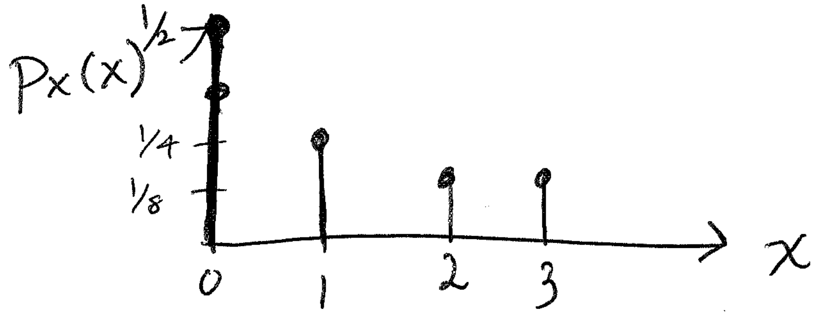
# (1) Fundamental Concepts



## DISCRETE DISTRIBUTIONS (PMF)

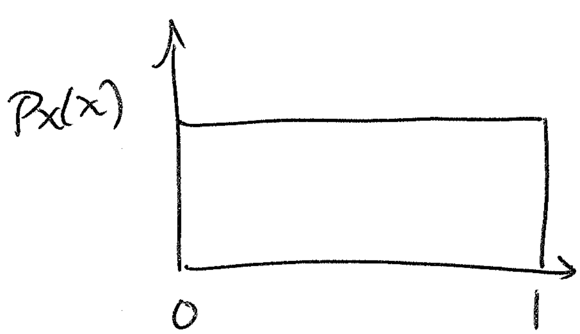
probability mass fn.




$$P_X(x) = \begin{cases} 1/2, & x = 0 \\ 1/4, & x = 1 \\ 1/8, & x = 2 \\ 1/8, & x = 3 \end{cases}$$

## CONTINUOUS DISTRIBUTIONS (PDF)

prob. density function




uniform over [0,1]

$$P_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwi} \end{cases}$$

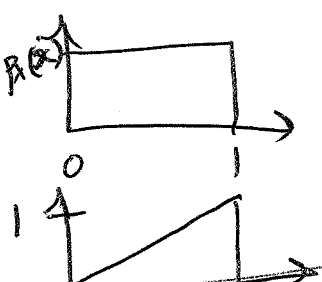
# Cumulative Distribution

$$F_X(c) = \int_{-\infty}^{c} p_X(x)\,dx = P[\infty \le X \le c]$$
$$= P[X \le c]$$

[accumulate probability (mass/density)
from $-\infty$ to some point $c$.]

$$\int_{-c}^{c} p_X(x)\,dx = \int_{-\infty}^{c} p_X(x)\,dx - \int_{-\infty}^{-c} p_X(x)\,dx$$

$$P[-c \le X \le c] = F_X(c) - (1 - F_X(-c))$$




pdf
↓
cdf

$$= 1 - F_X(c) + F_X(-c)$$

To ensure you have A proper CDF, $F_X(-\infty) = 0$, $F_X(\infty) = 1$.

Because

CDF $F_x(c) = \int_{-\infty}^{c} P_x(x)\, dx$

↳ is the integral of the pdf,

PDF $f_x(x) = \frac{d}{dx}\left(F_x(x)\right)$

↳ is the derivative of CDF.

## Bayes Rule + joint/condit. dist.

$$p(x|a) = \frac{p(x \cap a)}{p(a)}$$

$$= \frac{p(x,a)}{p(a)} \begin{array}{l} \longrightarrow \text{joint distribution} \\ \longrightarrow \text{marginal distribution} \end{array}$$

$$\Rightarrow \quad p(x|a)\,p(a) = p(x,a)$$

$$\Rightarrow \quad p(a|x)\,p(x) = p(x,a)$$

## marginalization

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} p_{X|Y}(x|y)\,p(y)\,dy$$

equivalently, $p_Y(y) = \int p_{X,Y}(x,y)\,dx$ .

## Moments / Expectation

$$E_x[g(x)] = \int_{-\infty}^{\infty} g(x)\, p_x(x)\, dx$$

x is distributed
w/ pdf $p_x(x)$.

## 1st Moment (mean)

$$E_x[X] = \int_{-\infty}^{\infty} x\, p_x(x)\, dx$$

## 2nd moment

$$E_x[X^2] = \int_{-\infty}^{\infty} x^2\, p_x(x)\, dx$$

## Variance

CONSTANT!
NOT RANDOM!

$$E_x[(X-E[X])^2] = E[X^2] - 2E[X\cdot E[X]] + E[X]^2$$

$$= \text{2nd moment} - 2E[X]^2 + E[X^2]$$

$$= \text{2nd moment} - (\text{mean})^2$$

EXAMPLE

$\boxed{X \text{ is dist. according to } p_X(x).}$

$$Y = aX + b.$$

$$E[Y] = E[aX+b]$$

$$= aE[X] + E[b]$$

CONSTANT!

$$= aE[X] + b$$

$$\text{cov}(X,Y) = E[(X-E[X])(Y-E[Y])]$$

$$= E[XY] - E[X]E[Y] - E[Y]E[X] + E[Y]E[X]$$

$$= E[XY] - E[X]E[Y]$$

$$= E[X(aX+b)] - E[X](aE[X]+b)$$

$$= E[aX^2] + b\cancel{E[X]} - aE[X]^2 - \cancel{bE[X]}$$

$$= aE[X^2] - aE[X]^2 \cancel{+b(E[} = a(E[X^2] - (E[X])^2)$$

# Conditional expectation



$E[X|Y]$ (evaluate argument wrt conditional dist)

$$= \int x\, p_{X|Y}(x|y)\, dx$$

## Expectation (2 VARIABLES)

$E[XY]$ evaluate argument wrt joint dist.

$$\|$$

$$E_{X,Y}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot p_{X,Y}(x,y)\, dx\, dy$$

# DERIVED DISTRIBUTIONS 

$X$ is distributed according to $p_X(x)$.

$Y = g(X)$.

$p_Y(y) = ?$ (derived distribution)

1) CDF of $Y$

$$F_Y(c) = P[Y \le c] = P[g(X) \le c]$$

$$= \int_{\{x \mid g(x) \le c\}} p_X(x)\, dx$$

2) Differentiate CDF of $Y$ to obtain pdf.

$$p_Y(y) = \frac{d}{dy}\left(F_Y(y)\right)$$

# Sums of 2 independent RVs 

$X, Y$ Are independent,
$X \sim U(0,1)$, $Y \sim U(0,1)$ (uniform)

$Z = X + Y$.

$p_Z(z) = ?$

$$p_Z(z) = \int_{-\infty}^{\infty} p_{X,Z}(x,z)\, dx$$

$$= \int_{-\infty}^{\infty} p_X(x) \underbrace{p_{Z|X}(z|x)}_{?}\, dx$$

Look @ CDF

$$F_{Z|X}(c) = P[Z \le c \mid X = x]$$

$$= P[Y + X \le c \mid X = x]$$

$$= P[Y \le c - x].$$

$$\Rightarrow F_{Z|X}(c) = P[Y \le c - x]$$

$$= F_Y(c-x)$$

$$\Rightarrow p_{Z|X}(z|x) = \underline{p_Y(\cancel{z}-x)}.$$

From before,

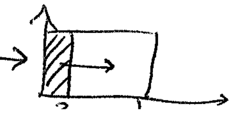
$$p_Z(z) = \int_{-\infty}^{\infty} p_X(x)\, p_{Z|X}(z|x)\, dx$$

$$= \int_{-\infty}^{\infty} p_X(x)\, p_Y(z-x)\, dx$$

<u>For our example</u> (need that $y = z - x \ge 0$ but less than 1)

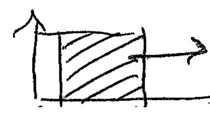
<u>For $0 \le z \le 1$</u>,

$$p_Z(z) = \int_0^z (1)(1)\, dx = z;$$



<u>For $1 \le z \le 2$</u>

$$p_Z(z) = 1 - \int_0^{z-1} (1)(1)\, dx$$



$$= 1 - (z-1) = 2 - z.$$

# Extra Credit (from QUIZ 1-2009)

(8/9)

A source transmits a signal $\theta$ with density,

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & \text{if } 0 \le \theta \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

The channel adds noise $N \sim \mathcal{N}(0, \sigma^2)$

Gaussian Shorthand

The detected signal is

$Y = \theta + N.$

(a) compute the conditional distribution of $Y$ given that $\theta = a$.

$$\Rightarrow Y = \underbrace{N + a}_{\text{scalar shift of the mean}} \Rightarrow P_{Y|\theta}(y \mid \theta = a) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-a)^2/2\sigma^2}$$

$\rightarrow N \sim \mathcal{N}(0, \sigma^2)$

$\rightarrow Y \sim \mathcal{N}(a, \sigma^2)$

9/9

(b) Compute $E[Y|\theta]$

$= E[N+\theta|\theta]$

$p_{Y|\theta}(y|\theta) =$ Gaussian w/ mean $(\theta)$, variance $(\sigma^2)$

$= \theta.$

(c) $E[Y] = \underbrace{E[N] + E[\theta]}_{\text{independence of } N,\, \theta} = 0 + \pi$

$= \pi.$