# ELEC 303 Midterm
# Solutions

October 14, 2010

1. For all the questions, you should precisely indicate your reasoning and show all your relevant work. Grades are based on your level of understanding as reflected by what you have written.
2. Write your solutions in this booklet. We will not grade the work not in this exam book.
3. You may give your answer in form of an arithmetic expression (sums products, ratios, factorials) of numbers that could be evaluated using a calculator.
4. This is a closed-book exam except for two double-sided, 8.5 by 11 formula sheets.
5. Be neat! If we cant read it, we cant grade it.

GOOD LUCK!!

1. **(10 points) False-Positive Puzzle**

A test for a rare disease has a false-positive rate of 10% (the person does not have the disease and tests positive) and a false-negative rate of 5% (the person has the disease and tests negative). A person drawn at random from a certain population has probability $p$ of having the disease.

Given that a person drawn at random tests positive for the disease, what is the probability that they actually have the disease?

A = person has disease

B = person tests positive

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$= \frac{0.9p}{0.9p + 0.1(1-p)}$$

$$= \frac{0.9p}{0.9p - 0.1p + 0.1}$$



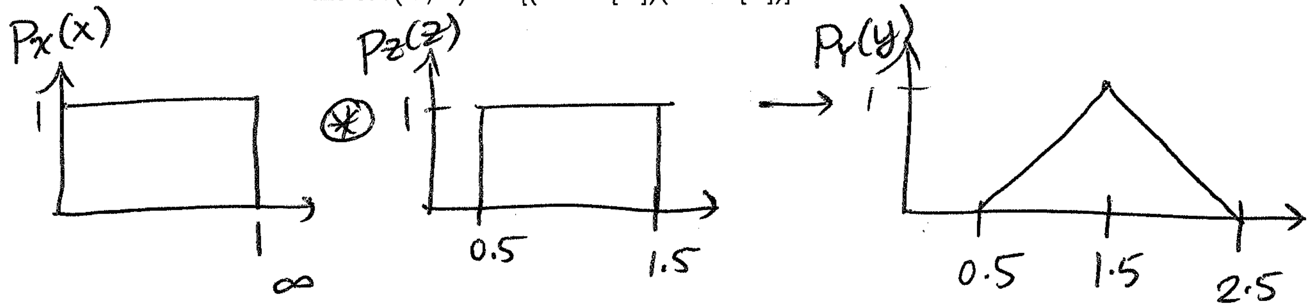
$$= \boxed{\frac{0.9p}{0.8p + 0.1}}.$$

2. **(30 points) Let $X$ and $Z$ be distributed uniformly over the intervals $[0, 1]$ and $[0.5, 1.5]$ respectively.**

   a. (15 pt) Compute the pdf of $Y = X + Z$, where $X$ and $Z$ are independent. Sketch a picture of the pdfs of $X$, $Y$, and $Z$ (make sure both axes are labeled).

   b. (15 pt) Let $Y = aX + b$, where $a$ and $b$ are scalars. Find $var[Y]$ and the $cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$.

(a)




$$P_Y(y) = \int_{-\infty}^{\infty} P_X(x)\, P_Z(y-x)\, dx$$

for $y \le 0.5$, $y \ge 2.5$ $\rightarrow$ $P_Y(y) = \boxed{0}$

for $0.5 \le y \le 1.5$

$$P_Y(y) = \int_{0.5}^{y} dx = x\Big|_{0.5}^{y} = \boxed{y - 0.5}.$$

for $1.5 \le y \le 2.5$

$$P_Y(y) = 1 - \int_{0}^{y-1.5} dx = 1 - x\Big|_{0}^{y-1.5} = 1 - (y - 1.5)$$

$$= \boxed{-y + 2.5}.$$

(b) $X \sim \mathcal{U}(0,1)$, $Y = aX + b$. ($VAR(Y) = ?$ $COV(X,Y) = ?$)

We need

$$E[X] = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$E[X^2] = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$E[Y] = E[aX+b] = aE[X] + b = \frac{a}{2} + b.$$

$$E[Y^2] = E[(aX+b)^2] = a^2E[X^2] + 2abE[X] + b^2.$$
$$= \frac{a^2}{3} + ab + b^2.$$

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$$VAR[Y] = E[Y^2] - E[Y]^2 = \frac{a^2}{3} + ab + b^2 - \left(\frac{a}{2} + b\right)^2$$
$$= \frac{a^2}{3} + \cancel{ab} + \cancel{b^2} - \frac{a^2}{4} - \cancel{ab} - \cancel{b^2} = \boxed{\frac{a^2}{12}}.$$

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$$COV[X,Y] = E[XY] - E[X]E[Y]$$
$$= E[X(aX+b)] - \frac{1}{2}\left(\frac{a}{2} + b\right)$$
$$= aE[X^2] + bE[X] - \frac{a}{4} - \frac{b}{2}$$
$$= \frac{a}{3} + \cancel{\frac{b}{2}} - \frac{a}{4} - \cancel{\frac{b}{2}} = \boxed{\frac{a}{12}}.$$

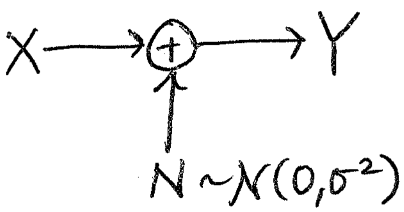
3. **(40 points) A source transmits a signal $X$ over a noisy channel, where $X = 0$ with probability $p$ and $X = 1$ with probability $1 - p$. The channel adds noise $N$ which we assume to be Gaussian (with mean 0 and variance $\sigma^2$) and independent of $X$. The detected signal is $Y = X + N$.**

   a. (5 points) Compute the conditional distribution, $p_{Y|X}(y|x)$.

   b. (10 points) Compute the conditional expectation, $\mathbb{E}[Y|X]$.

   c. (10 points) Compute the expected value of $Y$, $\mathbb{E}[Y]$.

   d. (15 points) At the detector, if $Y > c$, we declare that $X = 1$ was sent and if $Y \leq c$ we declare that $X = 0$ was sent. What is the probability of error, in terms of $c$? [Note: The integral of a Gaussian PDF does not have a closed form, simply write your answer in terms of the CDF, e.g., $\Phi(c) = P(Y \leq c)$.]




(a) $P_{Y|X}(y|x) = \boxed{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-x)^2/\sigma^2}}$ , $-\infty \leq y \leq \infty$

if $x$ is known, $Y \sim N(x, \sigma^2)$ (mean $= x$, VAR $= \sigma^2$)

(b) $E[Y \mid X = x] = E[N + x] = \cancel{E[N]} + x$

$= \boxed{x}$

(c) $E[Y] = E[X + N] = E[X] + E[N]$ (X, N are indep!)

$= 1 - p + 0 = \boxed{1 - p}$.

(d) $\hat{x}$ (estimate of $x$ @ detector): if $Y > c \rightarrow \hat{X} = 1$

$Y \leq c \rightarrow \hat{X} = 0$.

$P(\text{error}) = P[\hat{x} = 1 \mid X = 0] P[X = 0]$

$+ P[\hat{x} = 0 \mid X = 1] P[X = 1]$

continued

$$P(\text{error}) = (p)\, P[\hat{x}=1 \mid x=0]$$
$$+ (1-p)\, P[\hat{x}=0 \mid x=1]$$
$$= (p)\, P[y > c \mid x=0]$$
$$+ (1-p)\, P[y \le c \mid x=1]$$
$$= (p)\, P[N+0 > c]$$
$$+ (1-p)\, P[N+1 \le c]$$
$$= p\,(1 - P[N \le c]) + (1-p)\, P[N \le c-1]$$
$$= \boxed{p\,(1 - \Phi(c)) + (1-p)\, \Phi(c-1)}.$$