Review Quiz #2 elec303

Transform of $p_X(x)$:

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx}\, p_X(x)\, dx = E[e^{sX}]$$

Moment-generating function:

$$E[X^n] = \frac{d^n}{ds^n} M_X(s) \Big|_{s=0}$$

the n^{th} moment of X

Sums of independent RVs

$Z = X + Y$, (X, Y) are independent.

$$p_Z(z) = P(X+Y=z)$$
$$= \int_{-\infty}^{\infty} p_{X,Z}(x,z)\, dz = \int_{-\infty}^{\infty} p_X(x)\, p_Y(z-x)\, dx$$

convolution of $p_X(x)$ w/ $p_Y(y)$.

Alternatively,

$Z = X + Y$.

Compute the transform of Z.

$$M_Z(s) = E[e^{sZ}] = E[e^{s(X+Y)}] = E[e^{sX} e^{sY}]$$

$$= E[e^{sX}]\, E[e^{sY}]$$

$$= M_X(s) \cdot M_Y(s)$$

$p_Z(z) =$ inverse transform of $M_X(s) \cdot M_Y(s)$.

(section 4.5) Sum of a random number of independ. RVs

$Y = X_1 + \dots + X_n$, where n is random [takes on non-neg. integer values!] & X_i's are i.i.d (independent & identically distributed)

$$\boxed{p_Y(y) = ?}$$

Let $E[X_i] = \mu$
$VAR(X_i) = \sigma_x^2$ for all i.

The random variable
$Y = X_1 + \cdots + X_n$ is independent of N
and is therefore independent of $\{N = n\}$.

Hence,

$$E[Y \mid N=n] = E[X_1 + \cdots + X_N \mid N=n]$$
$$= E[X_1 + \cdots + X_n]$$
$$= n\,E[X] = n\mu.$$

This is true for any non-negative integer n, so

$$E[Y \mid N] = N\,E[X].$$

Using the law of iterated expectations, we obtain:

$$E[Y] = E[E[Y \mid N]] = E[N\,E[X]] = \underline{E[N]E[X]}.$$

$$\mathrm{VAR}(Y \mid N=n) = \mathrm{VAR}(X_1 + \cdots + X_N \mid N=n)$$

$$= \mathrm{VAR}(X_1 + \cdots + X_n)$$

$$= n \cdot \mathrm{Var}(X) = \underline{n\sigma_X^2}.$$

$$\Rightarrow \mathrm{VAR}(Y) = E[\mathrm{VAR}(Y \mid N)] + \mathrm{VAR}(E[Y \mid N])$$

$$= E[N\,\mathrm{VAR}(X)] + \mathrm{VAR}(N\,E[X])$$

$$= E[N]\,\mathrm{VAR}(X) + (E[X])^2\,\mathrm{VAR}(N).$$

Thus to find the transform of Y,

$$M_Y(s) = E[e^{sY}] = E[E[e^{sY} \mid N]]$$

$$E[e^{sX_1} \cdot e^{sX_2} \cdots e^{sX_N}] = (M_X(s))^N$$

$$= \underbrace{E[(M_X(s))^N]}_{?}$$

<u>using</u>

$$(M_X(s))^n = e^{\log(M_X(s))^n} =$$

$$= \underline{\exp(n \log(M_X(s)))}$$

We have that:

$$M_Y(s) = E_N[(M_X(s))^N]$$

↳ N is discrete & non-neg.

$$= \sum_{n=0}^{\infty} e^{n \log(M_X(s))} \cdot p_N(n)$$

and

$$M_N(s) = E[e^{sN}] = \sum e^{sn} p_N(n)$$

$\Downarrow$

implies that,

$$M_Y(s) = M_N(\log(M_X(s))) \xleftrightarrow[\text{inverse transf.}]{} P_Y(y)$$

Summary

for $Y = X_1 + \dots + X_N$, where $\underline{N}$ is random (integer & non-negative) & X_i's are i.i.d.

(1) $E[Y] = E[N]\, E[X]$.

(2) $VAR(Y) = E[N]\, VAR(X) + (E[X])^2 VAR(N)$.

(3) $M_Y(s) = M_N(\log M_X(s))$.

COVARIANCE & correlation

(1) Covariance

$$\mathrm{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$

$$= E[XY] - 2E[X]E[Y] + E[X]E[Y]$$

$$= E[XY] - E[X]E[Y].$$

if X, Y are independent?

$$E[XY] = E[X]E[Y]$$

$$\Rightarrow \mathrm{cov}(X,Y) = 0 \quad !$$

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(2) correlation COEFFICIENT

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{VAR}(X)\,\mathrm{VAR}(Y)}}$$

Variance of a sum of random variables

$$VAR(X_1 + X_2) = VAR(X_1) + VAR(X_2) + 2\,cov(X_1, X_2)$$

[see notes posted online]

* define $\tilde{X}_i = X_i - E[X_i]$ *

$$VAR\left(\sum_{i=1}^{n} X_i\right) = E\left[\left(\sum_{i=1}^{n} \tilde{X}_i\right)^2\right]$$

$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{X}_i \tilde{X}_j\right]$$

$$= \sum_{i=1}^{n}\sum_{i=1}^{n} E[\tilde{X}_i \tilde{X}_j]$$

$$= \sum_{i=1}^{n} E[\tilde{X}_i^2] + \sum_{\{(i,j),\, i \neq j\}} E[\tilde{X}_i \tilde{X}_j]$$

$$= \sum_{i=1}^{n} VAR(X_i) + \sum_{\substack{i,j \\ i \neq j}} cov(X_i, X_j).$$

PROBLEM 27 (pg. 253)

We toss a biased coin ($P(\text{head}) = q$) n times, where q is a value of a RV Q w/ mean $= \mu$, VAR $= \sigma^2$.

Let X_i be a Bernoulli RV that models the outcome of the i^{th} toss ($X_i = 1$ if i^{th} toss is a head).
Assume that $X_1, \ldots, X_n$ are conditionally independent, given $Q = q$. Let $X = \sum_{i=1}^{n} X_i$ be the # of heads in n tosses.

(a) Use the law of iterated expectations to find $E[X_i]$ and $E[X]$.

⊛ $E[X_i \mid Q] = Q$ [expectation of a Bernoulli w/ parameter Q]

$$E[X_i \mid Q] = \sum_{i=0}^{1} x_i \, p_{X_i}(x_i)$$
$$= (0)(1-Q) + (1)Q$$
$$= \underline{Q}.$$

$$E[X_i] = E[E[X_i \mid Q]]$$
$$= E[Q] = \underline{\mu}.$$

Since $X = X_1 + \cdots + X_n$

$$E[X] = E[X_1] + \cdots + E[X_n]$$
$$= \underline{n\mu}.$$

(b) Find $cov(X_i, X_j)$. Are $X_1, \ldots, X_n$ independ?

(X_i, X_j assumed to be condit independ!!)

* For $i \neq j$,

$$E[X_i X_j | a] = E[X_i | a] \cdot E[X_j | a]$$
$$= \underline{a^2}.$$

$$E[X_i X_j] = E_a[E[X_i X_j | a]]$$
$$= \underline{E[a^2]}.$$

Thus,

$$cov(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j]$$
$$= E[a^2] - \mu^2 = \underline{\sigma^2}.$$

Since $cov(X_i, X_j) \neq 0 \Rightarrow X_i, X_j$ Are NOT independent!!

For $i=j$ $\rightarrow$ $X_i^2 = X_i$

$$VAR(X_i) = E[X_i^2] - (E[X_i])^2$$

$$= E[X_i] - (E[X_i])^2$$

$$= \underline{\mu - \mu^2}.$$

(c) Use the law of total variance to find $VAR(X)$. Verify your answer using the cov in part (b).

$$VAR(X) = E[VAR(X \mid Q)] + VAR(E[X \mid Q])$$

$$= E[VAR(X_1 + \cdots + X_n \mid Q)] + VAR(E[X_1 + \cdots + X_n \mid Q])$$

$$= \underbrace{E[nQ(1-Q)]}_{\text{condit. independ.}} + \underbrace{VAR(nQ)}_{VAR(aX) = a^2 VAR(X)}$$

$$= n\,E[Q - Q^2] + n^2 VAR(Q)$$

$$= n(\mu - \mu^2 - \sigma^2) + n^2\sigma^2 = \underline{n(\mu - \mu^2) + n(n-1)\sigma^2}.$$

To verify the results from part (b),

$$VAR(X) = VAR(X_1 + \cdots X_n)$$

$$= \sum_{i=1}^{n} VAR(X_i) + \sum_{i \neq j} COV(X_i, X_j)$$

$$= n\,VAR(X_i) + n(n-1)\,COV(X_1, X_2)$$

$$= n(\mu - \mu^2) + n(n-1)\sigma^2 .$$

Same as before !!

Review –

Normal Approximation Based on CLT

Let $S_n = X_1 + \cdots + X_N$,

where X_i's are i.i.d w/ mean $= \mu$
VAR $= \sigma^2$.

$$P(S_n \le c)$$

$$= P\left(\frac{S_n - n\mu}{\sigma \cdot \sqrt{n}} \le \frac{c - n\mu}{\sigma\sqrt{n}}\right)$$

$$= P\left(Z_n \le \frac{c - n\mu}{\sigma\sqrt{n}}\right)$$

$$\approx \underline{\Phi}\left(\frac{c - n\mu}{\sigma\sqrt{n}}\right).$$

↳ CDF of a STANDARD normal w/ mean $= 0$
VAR $= 1$.

A note on Gaussian CDFs.

$Y \sim N(\mu, \sigma^2)$, $X \sim \underbrace{N(0,1)}_{\text{STANDARD NORMAL}}$

$\underline{Y = \sigma X + \mu}$. (property of gaussians)

Thus for

$$\Phi(c) = \int_{-\infty}^{c} e^{-x^2/2}\, dx = P(X \leq c),$$

we can express the CDF of AN arbitrary gaussian in terms of the STANDARD NORMAL CDF

$$P(Y \leq c) =$$

$$= P(\sigma X + \mu \leq c) = P\left(X \leq \frac{c-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{c-\mu}{\sigma}\right).$$