Transform of $p_x(x)$:

$$M_x(s) = \int_{-\infty}^{\infty} e^{sx} p_x(x) \, dx = E[e^{sx}]$$

Moment-generating function:

$$E[X^n] = \left. \frac{d^n}{ds^n} M_x(s) \right|_{s=0}$$

the $n^{th}$ moment of $X$

Sums of Independent RVs

$Z = X + Y$, $(X,Y)$ are independent.

$$P_z(z) = P(X + Y = z) = \int_{-\infty}^{\infty} P_{X,Z}(x,z) \, dx = \int_{-\infty}^{\infty} p_x(x) \, p_y(z-x) \, dx$$

Convolution of $p_x(x)$ w/ $p_y(y)$.
Alternatively,

\[ Z = X + Y. \]

Compute the transform of \( Z \).

\[
M_Z(s) = E[e^{sZ}] = E[e^{s(X+Y)}] = E[e^{sX}e^{sY}]
= E[e^{sX}]E[e^{sY}]
= M_X(s) \cdot M_Y(s)
\]

\( P_Z(z) = \text{inverse transform of } M_X(s) \cdot M_Y(s). \)

---

**Section 4.5** Sum of a Random Number of Independent RVs

\[ Y = X_1 + \ldots + X_n \]

\[ P_Y(y) = ? \]

Where \( n \) is random (takes on non-negative integer values!)

\&

\( X_i \)'s are i.i.d (independent & identical distributed)
Let \( E[X_i] = \mu \) for all \( i \).
\[ \text{VAR}(X_i) = \sigma_x^2 \]

The random variable
\[ Y = X_1 + \cdots + X_n \]
is independent of \( N \) and therefore independent of \( \exists N = n \).

Hence,
\[
E[Y \mid N=n] = E[X_1 + \cdots + X_n \mid N=n]
\]
\[
= E[X_1 + \cdots + X_n]
\]
\[
= n \ E[X] = n \mu .
\]

This is true for any non-negative integer \( n \), so
\[
E[Y \mid N] = N \ E[X].
\]

Using the law of iterated expectations, we obtain:
\[
E[Y] = E[E[Y \mid N]] = E[N E[X]] = E[N] E[X].
\]
\[
\text{VAR}(Y \mid N=n) = \text{VAR}(X_1 + \cdots + X_N \mid N=n) \quad 4
\]
\[
= \text{VAR}(X_1 + \cdots + X_n)
\]
\[
= n \cdot \text{VAR}(X) = n \sigma_x^2.
\]
\[
\Rightarrow \text{VAR}(Y) = E[\text{VAR}(Y \mid N)] + \text{VAR}(E[Y \mid N])
\]
\[
= E[N \text{VAR}(X)] + \text{VAR}(N E[X])
\]
\[
= E[N] \text{VAR}(X) + (E[X])^2 \text{VAR}(N).
\]

Thus to find the transform of \(Y\),
\[
M_Y(s) = E[e^{sY}] = E[E[e^{sY} \mid N]] = E[e^{sX_1}e^{sX_2} \cdots e^{sX_N}]
\]
\[
= E[(M_X(s))^N]
\]
\[
\text{Using } (M_X(s))^n = e^{\log(M_X(s))^n} = e^{n \log(M_X(s))}
\]
\[
= \exp(n \log(M_X(s)))
\]
We have that:

\[ M_Y(s) = E_n \left[ (M_X(s))^N \right] \]

\[ = \sum_{n=0}^{\infty} e^{-n \log(M_X(s))} \cdot p_N(n) \]

\[ \text{N is discrete & non-neg.} \]

and

\[ M_N(s) = E[e^{sN}] = \sum e^{sn} p_N(n) \]

\[ \downarrow \]

implies that,

\[ M_Y(s) = M_N(\log(M_X(s))) \xrightarrow{\text{inverse transf.}} P_Y(y) \]

(Summary)

For \( Y = X_1 + \ldots + X_N \), where \( N \) is random [integer & non-negative] & \( X_i's \) are i.i.d.

(1) \( E[Y] = E[N] E[X] \).

(2) \( \text{VAR}(Y) = E[N] \text{VAR}(X) + (E[X])^2 \text{VAR}(N) \).

(3) \( M_Y(s) = M_N(\log M_X(s)) \).
(1) Covariance

\[
\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]
\]

\[
\]

\[
= E[XY] - E[X]E[Y].
\]

If \( X, Y \) are independent?

\[
E[XY] = E[X]E[Y]
\]

\[
\Rightarrow \text{Cov}(X, Y) = 0.
\]

(2) Correlation Coefficient

\[
\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}
\]
Variance of a sum of random variables

\[ \text{VAR}(X_1 + X_2) = \text{VAR}(X_1) + \text{VAR}(X_2) + 2 \text{COV}(X_1, X_2) \]

[see notes posted online]

* define \( \hat{X}_i = X_i - E[X_i] \) *

\[ \text{VAR} \left( \sum_{i=1}^{n} X_i \right) = E \left[ \left( \sum_{i=1}^{n} \hat{X}_i \right)^2 \right] \]

\[ = E \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{X}_i \hat{X}_j \right] \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{n} E[\hat{X}_i \hat{X}_j] \]

\[ = \sum_{i=1}^{n} E[X_i^2] + \sum_{\substack{i \neq j \\{i,j\} \neq \emptyset}} E[\hat{X}_i \hat{X}_j] \]

\[ = \sum_{i=1}^{n} \text{VAR}(X_i) + \sum_{i \neq j} \text{COV}(X_i, X_j) \]
We toss a fair coin \((P(\text{head}) = \frac{1}{2})\) \(n\) times, where \(q\) is a value of a RV \(\alpha\) with mean \(\mu\), \(\text{VAR} = \sigma^2\).

Let \(X_i\) be a Bernoulli RV that models the outcome of the \(i\text{th}\) toss \((X_i = 1 \text{ if } i\text{th toss is a head})\).

Assume that \(X_1, \ldots, X_n\) are conditionally independent, given \(\alpha = q\). Let \(X = \sum_{i=1}^{n} X_i\) be the number of heads in \(n\) tosses.

(a) Use the law of iterated expectations to find \(E[X_i]\) and \(E[X]\).

\[ E[X_i | \alpha] = \alpha \]
\[ E[X_i] = E[E[X_i | \alpha]] = E[\alpha] = \mu. \]

\[ E[X_i | \alpha] = \sum_{x_i=0}^{1} x_i P_{X_i}(x_i) \]
\[ = (0)(1-\alpha) + (1)\alpha \]
\[ = \alpha. \]
Since \( X = X_1 + \ldots + X_n \)

\[
E[X] = E[X_1] + \ldots + E[X_n] = n \mu.
\]

(b) Find \( \text{cov}(X_i, X_j) \). Are \( X_1, \ldots, X_n \) independent?

\[
\text{For } i \neq j, \quad (X_i, X_j \text{ assumed to be conditionally independent})
\]

\[
E[X_i X_j | \alpha] = E[X_i | \alpha] \cdot E[X_j | \alpha] = \alpha^2.
\]

\[
E[X_i X_j] = E[E[X_i X_j | \alpha]] = E[\alpha^2].
\]

Thus,

\[
\text{cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j] = E[\alpha^2] - \mu^2 = \sigma^2.
\]

Since \( \text{cov}(X_i, X_j) \neq 0 \Rightarrow X_i, X_j \text{ are not independent}!! \)
For $i=j \quad \rightarrow \quad X_i^2 = X_i$

\[
\text{VAR}(X_i) = E[X_i^2] - (E[X_i])^2
\]

\[
= E[X_i] - (E[X_i])^2
\]

\[
= \mu - \mu^2.
\]

(c) Use the law of total variance to find $\text{VAR}(X)$. Verify your answer using the cov in part (b).

\[
\text{VAR}(X) = E[\text{VAR}(X|\alpha)] + \text{VAR}(E[X|\alpha])
\]

\[
= E[\text{VAR}(X_1 + \cdots + X_n|\alpha)] + \text{VAR}(E[X_1 + \cdots + X_n|\alpha])
\]

\[
= E[n\alpha(1-\alpha)] + \text{VAR}(n\alpha)
\]

\[
= n E[\alpha - \alpha^2] + n^2 \text{VAR}(\alpha)
\]

\[
= n (\mu - \mu^2 - \sigma^2) + n^2 \sigma^2
\]

\[
= n(\mu-M^2) + n(n-1)\sigma^2.
\]
To verify the results from part (b),

\[ \text{VAR}(x) = \text{VAR}(x_1 + \cdots + x_n) \]

\[ = \sum_{i=1}^{n} \text{VAR}(x_i) + \sum_{i \neq j} \text{cov}(x_i, x_j) \]

\[ = n \text{VAR}(x_1) + n(n-1) \text{cov}(x_1, x_2) \]

\[ = n(m - m^2) + n(n-1) \sigma^2 \]

Same as before!!
Normal Approximation Based on CLT

Let $S_n = X_1 + \cdots + X_N$, where $X_i$'s are i.i.d. w/ mean $\mu$, $\text{VAR} = \sigma^2$.

$$P(S_n \leq c)$$

$$= P\left( \frac{S_n - n\mu}{\sigma \sqrt{n}} \leq \frac{c - n\mu}{\sigma \sqrt{n}} \right)$$

$$= P(Z_n \leq \frac{c - n\mu}{\sigma \sqrt{n}})$$

$$\approx \Phi\left( \frac{c - n\mu}{\sigma \sqrt{n}} \right).$$

Let $\Phi$ be the CDF of a **standard normal** w/ mean $0$, $\text{VAR} = 1$. 
A note on Gaussian CDFs.

\[ Y \sim N(\mu, \sigma^2), \quad X \sim N(0,1) \]

\[ Y = \sigma X + \mu \]  \quad \text{(Property of Gaussians)}

Thus for

\[ \Phi(c) = \int_{-\infty}^{c} e^{-x^2/2} \, dx = P(X \leq c), \]

we can express the CDF of an arbitrary Gaussian in terms of the Standard Normal CDF

\[ P(Y \leq c) = \]

\[ = P(\sigma X + \mu \leq c) = P(X \leq \frac{c - \mu}{\sigma}) \]

\[ = \Phi \left( \frac{c - \mu}{\sigma} \right). \]