Final Review (PART ONE)

RANDOM PROCESSES / VECTORS:

Some definitions,

$\boxed{\text{autocorrelation}}$ and looks @ similarity between original signal/process and a shifted version

Continuous-time $X(t)$	$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$
	$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t_1)X(t_2) f_{X_1,X_2}(x_1,x_2)\, dx_1 dx_2$
Discrete-time $\underline{X} \in \mathbb{R}^N$	$R_{\underline{X}} = E[XX^T]$
	where $XX^T \in \mathbb{R}^{N \times N}$

Cross-correlation → similarity between shifted version of 2 different processes.

Continuous time + $X(t)$, $Y(t)$ $\quad R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$

DISCRETE → $R_{\underline{X}\underline{Y}} = E[XY^T]$.

Properties of AUTOCORRELATION

1) $R_X(\tau=0) = E[X(t)\,X(t)]$

$$= E[X_t^2]$$

Second-moment (POWER)

2) maximal @ origin

$$|R_X(\tau)| \leq E[X_t^2]$$

3) <u>REAL</u>.

Covariance

$$\mathrm{Cov}(X,Y) = \underset{\downarrow}{E[XY^T]} - \underbrace{\mu_X \mu_Y^T}$$

$\begin{bmatrix} N\times N \\ \text{matrix} \\ X \in \mathbb{R}^N \\ Y \in \mathbb{R}^N \end{bmatrix}$ $\begin{bmatrix} N\times N \text{ matrix} \\ \mu_X \in \mathbb{R}^N \\ \mu_Y \in \mathbb{R}^N \end{bmatrix}$

$$= R_{XY} - \mu_X \mu_Y^T .$$

* If X, Y are independent,

$E[X(t)\,Y(t)] = E[X(t)]\,E[Y(t)]$.

If X, Y are uncorrelated,

$\mathrm{Cov}(X,Y) = 0$.

$\Rightarrow$ If X, Y are zero mean, $E[XY^T] = R_{XY} = 0$.

WIDE-SENSE STATIONARY PROCESSES

1) $E[X(t)]$ = constant.

2) $R_X(t_1, t_2)$ only depends on time difference $t_1 - t_2 = \tau$

$\Rightarrow R_X(\tau) = E[X(t)\,X(t-\tau)]$

can be simplified such that all dependence on $\underline{t}$ is REMOVED.

A consequence of WSS,

$$\boxed{R_X(\tau) \xleftrightarrow{\mathcal{F}} S_X(f)}$$

Wiener-Khinchin Theorem

POWER SPECTRAL DENSITY

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f\tau}\, df$$

evaluating $R_X(\tau)$ @ zero,

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, e^0 \, df$$

$$= \int_{-\infty}^{\infty} S_X(f)\, df$$

$$= \text{POWER!}$$

Properties of PSDs

valid power spectral densities must be,

→ non-negative everywhere

→ even (because it is a Fourier transf. of R_X which is REAL)

→ REAL-VALUED.

Linear Filtering of Random Processes

$$X(t) \longrightarrow \boxed{H} \longrightarrow Y(t)$$

Two questions of interest

* Analysis – given X, H, what will Y look like?

* Design – given X and a desired Y, what should H be?

analysis

Expectation → Mean

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(t-\alpha)\, X(\alpha)\, d\alpha\right]$$

$$= \int_{-\infty}^{\infty} h(t-\alpha)\, \underbrace{E[X(\alpha)]}_{\substack{\| \\ m_X(t)}}\, d\alpha$$

$m_X(t)$: mean over time

= filtered version of mean!

Autocorrelation of Y?

$$R_Y(t_1,t_2) = E[Y(t_1)\,Y(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1-\alpha)\,h(t_2-\beta)\,E[X(\alpha)\,X(\beta)]\,d\alpha\,d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1-\alpha)\,h(t_2-\beta)\,R_X(\alpha,\beta)\,d\alpha\,d\beta$$

If $X(t)$ is W.S.S, change of var. $\left(\begin{array}{l} \eta = t_1-\alpha,\ \gamma = \alpha-\beta \\ \alpha = t_1-\eta,\ \beta = \alpha-\gamma \end{array}\right)$

$$R_Y(t_1,t_2) = \iint h(\eta)\,h(t_2-t_1+\eta+\gamma)\,R_X(\gamma)\,d\gamma\,d\eta$$

$\tau = t_1-t_2$

(autocorr only depends on $\alpha-\beta=\gamma$.)

$$R_Y(\tau) = \iint h(\eta)\,h(\tau+\eta+\gamma)\,R_X(\gamma)\,d\gamma\,d\eta$$

POWER SPECTRAL DENSITY

we had that,

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\eta)\, \underbrace{h(\tau+\eta+\gamma)}\, R_X(\gamma)\, d\gamma\, d\eta$$

only term dependent on τ (h shifted by $\eta+\gamma$) (transform is $H(f)e^{j2\pi f(\eta+\gamma)}$)

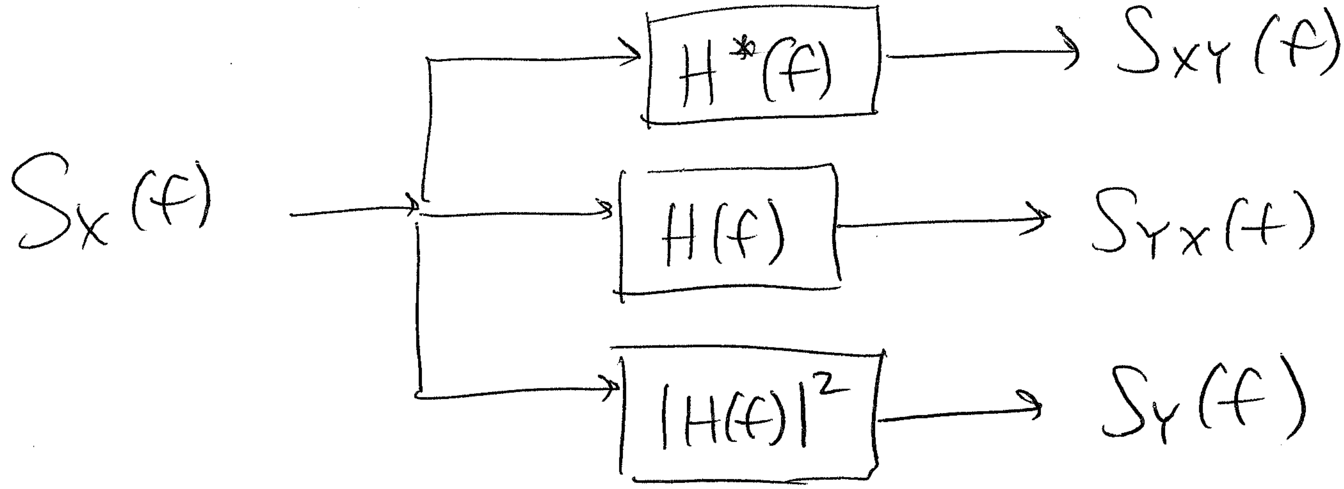
$$\mathcal{F} \updownarrow$$

$$S_Y(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\eta)\, H(f)\, e^{j2\pi f(\eta+\gamma)}\, R_X(\gamma)\, d\gamma\, d\eta$$

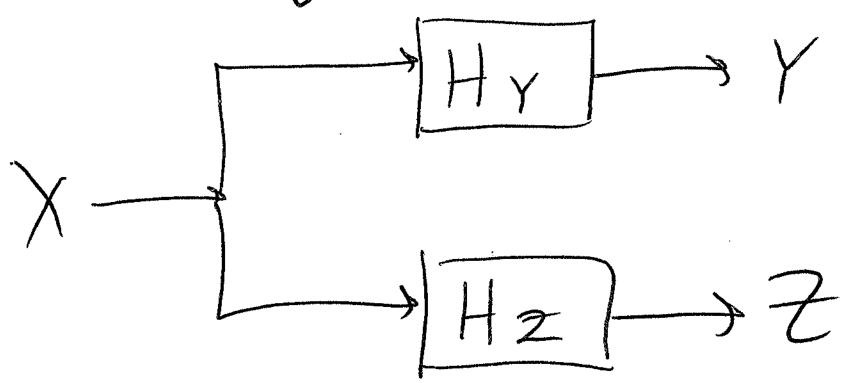
$$= H(f) \underbrace{\left(\int_{-\infty}^{\infty} h(\eta) e^{j2\pi f\eta}\, d\eta\right)}_{H^*(f)} \underbrace{\left(\int_{-\infty}^{\infty} R_X(\gamma) e^{j2\pi f\gamma}\, d\gamma\right)}_{S_X^*(f) = S_X(f)}$$

because S_X is REAL, even, symmet.

$$= H(f)\, H^*(f)\, S_X(f) = \underline{|H(f)|^2 S_X(f)}\,.$$



CROSS-POWER SPECTRAL DENSITY



$$S_{YZ}(f) = S_X(f)\, H_Y^*(f)\, H_Z(f)$$

$$S_{ZY}(f) = S_X(f)\, H_Z^*(f)\, H_Y(f)$$

Now, the SECOND QUESTION,

DESIGN ...

$$\boxed{X \to \overset{?}{H} \to Y}$$

How do you design a filter for a desired output for Y ?

Example (whitening Filters)

(White Processes) if $X(t)$ is white,

1) $S_X(\ )$ is constant

for continuous-time processes, $S_X(\omega) = C$ for $-\infty \le \omega \le \infty$

discrete-time, $S_X(f) = C$ for $-\pi \le \omega \le \pi$

2) $R_X(\tau) = \delta(\tau)$ (cont-time)

$$\underbrace{R_X[n] = \delta[n]}$$

X is only correlated with itself @ ZERO time lag (all other shifts ARE uncorrel.)

Assume

$X(t)$ is white $\Rightarrow S_X(f) = \frac{N_0}{2}$.

$$X(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

$$S_Y(f) = |H(f)|^2 S_X(f) = \frac{N_0}{2} |H(f)|^2$$

if our output PSD is $\frac{N_0}{2} |H(f)|^2$, and we wish to find a filter to whiten Y,

we must find $H^{-1}(f)$.

$$X(t) \longrightarrow \boxed{H} \xrightarrow{\ Y\ } \boxed{H^{-1}} \longrightarrow \hat{X}(t)$$

should be white

What properties must H have for H^{-1} to exist?

→ can't be zero (otherwise $\frac{1}{0} \to \infty$)

→ must be minimum-phase

why? Lets see...

Lets look @ an example,

Difference EQ

$$\boxed{Y[n] = \sum_{k=1}^{A} a_k Y[n-k] + \sum_{k=0}^{B} X[n-k] b_k}$$

$$\Big\downarrow z$$

$$Y(z)\left(1 - \sum_{k=1}^{A} a_k z^{-k}\right) = X(z)\left(1 + \sum_{k=1}^{B} b_k z^{-k}\right)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \sum_{k=1}^{B} b_k z^{-k}}{1 - \sum_{k=1}^{A} a_k z^{-k}}$$

and $\underbrace{H(\omega)}_{\text{Fourier transform}} = H(z)\Big|_{z = e^{j\omega}}$

Lets say that (homework 8, PROB. 2)

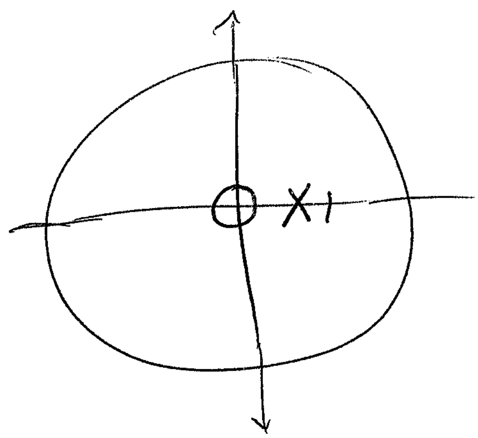
$$Y[n] = X[n] + \frac{1}{3} Y[n-1]$$

$$\Big\downarrow z$$

$$Y(z)\left(1 - \frac{1}{3} z^{-1}\right) = X(z) \Rightarrow H(z) = \frac{1}{1 - \frac{1}{3} z^{-1}}.$$

$$H(z) = \frac{1}{1-\frac{1}{3}z^{-1}} \cdot \frac{z}{z} = \frac{z}{z-\frac{1}{3}}.$$

The poles & zeros would be,



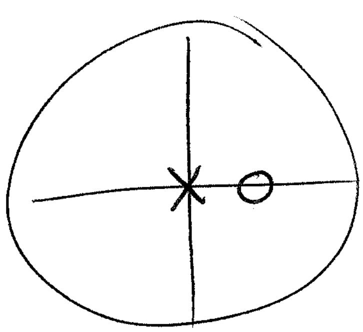
zero @ 0

pole @ $\frac{1}{3}$

Hence the inverse filter would be,

$$H^{-1}(z) = \frac{1}{H(z)} = 1 - \frac{1}{3}z^{-1} = \frac{z-\frac{1}{3}}{z}$$

poles and zeros



So every pole of H is replaced w/ a zero & every zero is replaced by a pole.

What this says is,

if H has all of its zeros and poles INSIDE the unit circle,
(zeros in circle $\Rightarrow$ min. phase)
(poles in circle $\Rightarrow$ stable filter)

WE can find an INVERSE filter that is stable.

for our example

$$H'(z) = \frac{z - \frac{1}{3}}{z} = 1 - \frac{1}{3}z^{-1}$$

$$\Big\downarrow z^{-1}$$

$$h[n] = \delta[n] - \frac{1}{3}\delta[n-1].$$