(Review - Part Two)

Hypothesis Testing

setup

Hypothesis ~ H_1
Null hypothesis ~ H_0

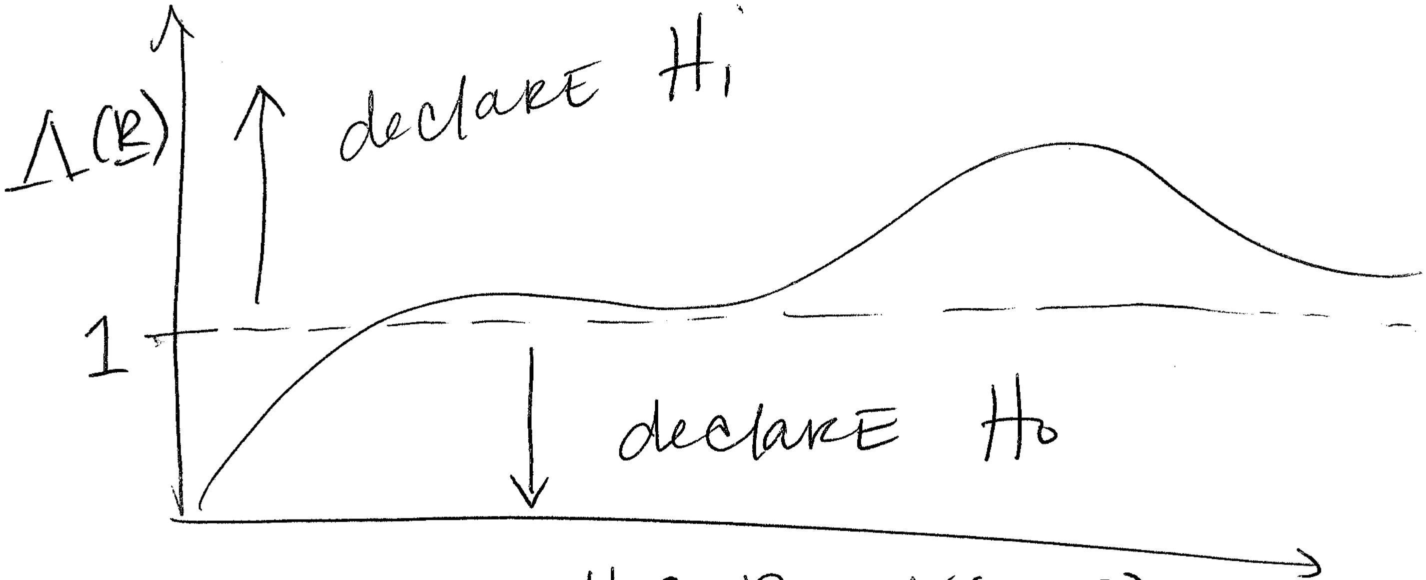
$p(H_1) = \pi_1$, $p(H_0) = \pi_0$.
observations → $\underline{R}$.

If you have NO prior knowledge that one hypothesis is more likely than the other,

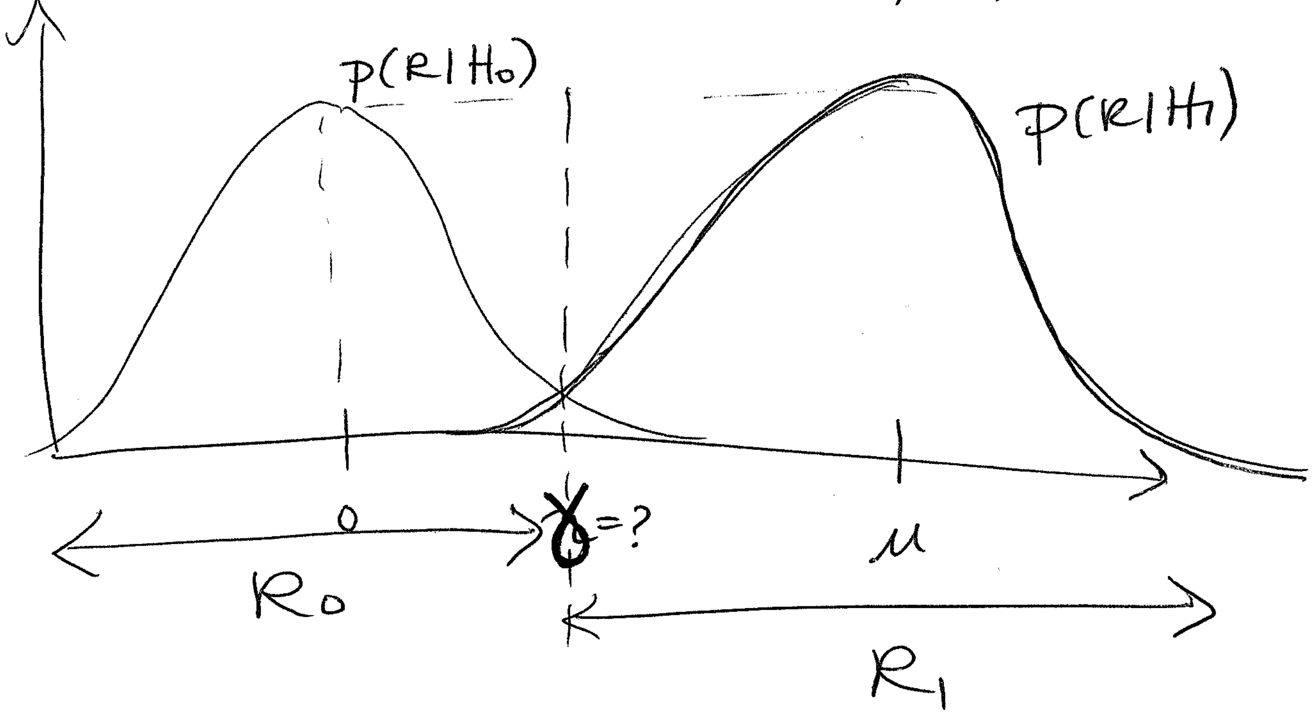
Form the Likelihood Ratio $\Lambda(\underline{R})$,

$$\left[\Lambda(\underline{R}) = \frac{p(R \mid H_1)}{p(R \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 .\right]$$

if the LRatio $\Lambda(\underline{R})$ is greater than one we declare R was generated under H_1, otherwise ⇒ H_0.



EXAMPLE: H_0: $R \sim N(0, \sigma^2)$
H_1: $R \sim N(\mu, \sigma^2)$



if R lies in R_0, declare H_0 $(R < \gamma)$
if R lies in R_1, declare H_1 $(R > \gamma)$

What should γ be?

What is γ?

For our example $\left(\begin{array}{l} H_1: R \sim N(\mu, \sigma^2) \\ H_0: R \sim N(0, \sigma^2) \end{array} \right)$

$$p(R \mid H_1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$p(R \mid H_0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x^2)}$$

$$\Lambda(R) = \frac{e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}{e^{-\frac{1}{2\sigma^2}x^2}} = e^{-\frac{1}{2\sigma^2}(\cancel{x^2} - 2\mu x + \mu^2 - \cancel{x^2})}$$

$$= e^{-\frac{1}{2\sigma^2}(\mu^2 - 2\mu x)}$$

Set $R = X$.

if our usual comparison is for $\Lambda(R) \gtrless 1$, we can simplify $\Lambda(R)$ to get a sufficient statistic for R,

$$\Lambda(R) = e^{-\frac{1}{2\sigma^2}(\mu^2 - 2\mu R)} \gtrless 1$$

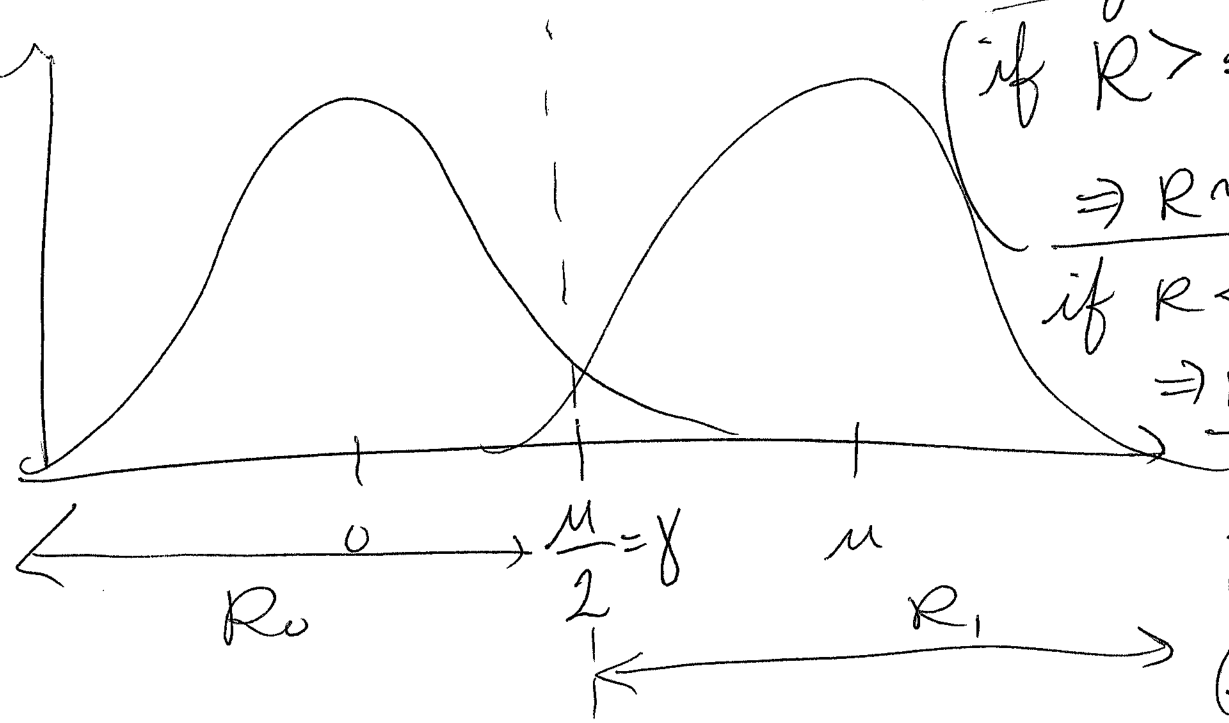
$$-\frac{1}{2\sigma^2}(\mu^2 - 2\mu R) \gtrless \ln(1)$$

$$-\frac{1}{2\sigma^2}\mu^2 + \frac{\mu R}{\sigma^2} \gtrless \ln(1) = 0$$

$$\Rightarrow \frac{\mu R}{\sigma^2} \gtrless \frac{\mu^2}{2\sigma^2}$$

$$\Rightarrow R \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\mu}{2} = \gamma$$

Looking @ our graph from before,



we say that

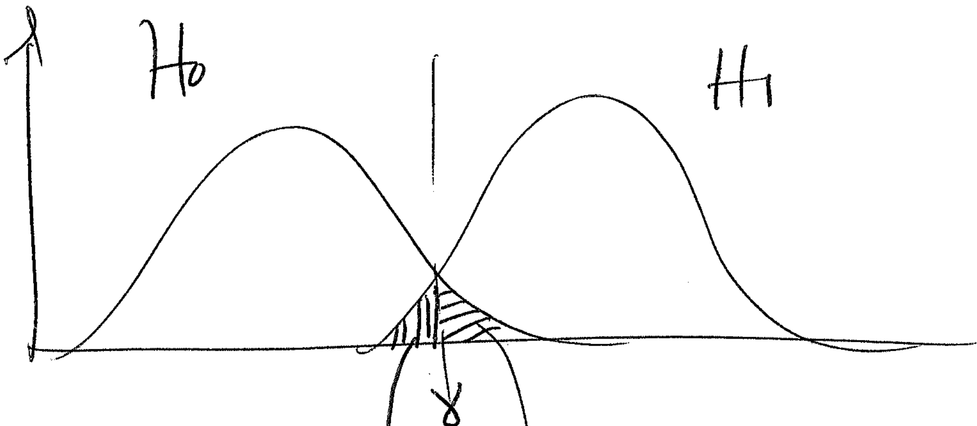
if $R > \frac{\mu}{2}$ $\Rightarrow R \sim H_1$

if $R < \frac{\mu}{2}$ $\Rightarrow R \sim H_0$

$\Downarrow$

if $R = \frac{\mu}{2}$ (flip a coin

This will minimize the total error,



$P(R < \gamma \mid H_1) \Rightarrow$ miss probability (say H_0 when it's actually H_1)

$P(R > \gamma \mid H_0)$

$\Rightarrow$ false alarm probability (say H_1 when its actually H_0)

What if the cost of a FALSE ALARM is higher than missing H_1?

(e.g., saying H_1 when H_0 is TRUE → FALSE ALARM
is WORSE than
saying H_0 when H_1 is TRUE)

(cost for FALSE ALARM)

C_{10} : cost of deciding H_1 when H_0 is true

C_{10} : cost of deciding H_0 when H_1 is TRUE MISS

(cost for misses)

INSTEAD

$$p(R \mid H_1)\, C_{01} \underset{H_0}{\overset{H_1}{\gtrless}} p(R \mid H_0)\, C_{10}$$

Before $\Lambda(R) = \dfrac{p(R \mid H_1)}{p(R \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$ ← back to original CASE

Now, $\Lambda(R) = \dfrac{p(R \mid H_1)}{p(R \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \dfrac{C_{10}}{C_{01}}$ (if equal)

Going back to the gaussian example from before, (with costs)

$$\frac{1}{2\sigma^2}\left(2\mu R - \mu^2\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{C_{10}}{C_{01}}\right)$$

$$\Rightarrow 2\mu R \underset{H_0}{\overset{H_1}{\gtrless}} 2\sigma^2\left(\ln\left(\frac{C_{10}}{C_{01}}\right)\right) + \mu^2$$

$$\Rightarrow R \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\frac{\sigma^2}{\mu}\left(\ln\left(\frac{C_{10}}{C_{01}}\right)\right)} + \frac{\mu}{2} = \gamma$$

this was zero before, so $\gamma = \frac{\mu}{2}$.

If false alarms are $2\times$ worse than misses, $\underline{C_{10} = 2C_{01}}$.

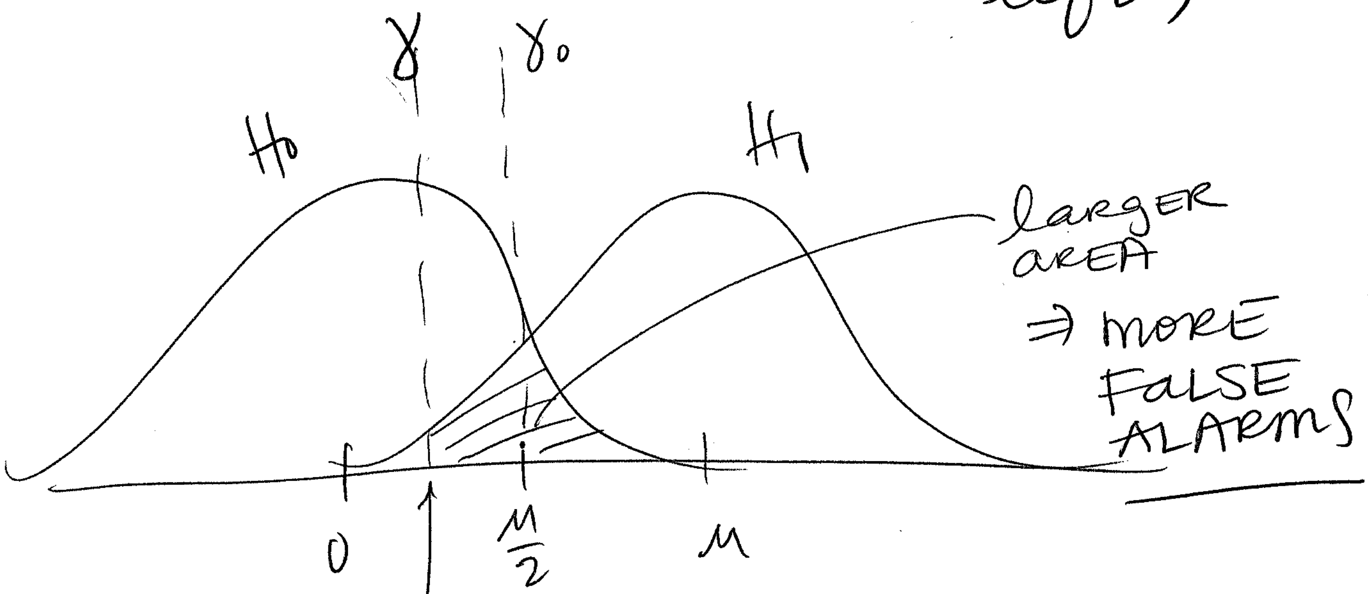
What about the converse?

$$C_{01} = 2C_{10}$$

(misses are 2× worse than false alarms)

$$R \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\mu}{2} + \frac{\sigma^2}{\mu}\left(\ln\left(\tfrac{1}{2}\right)\right) = \gamma$$

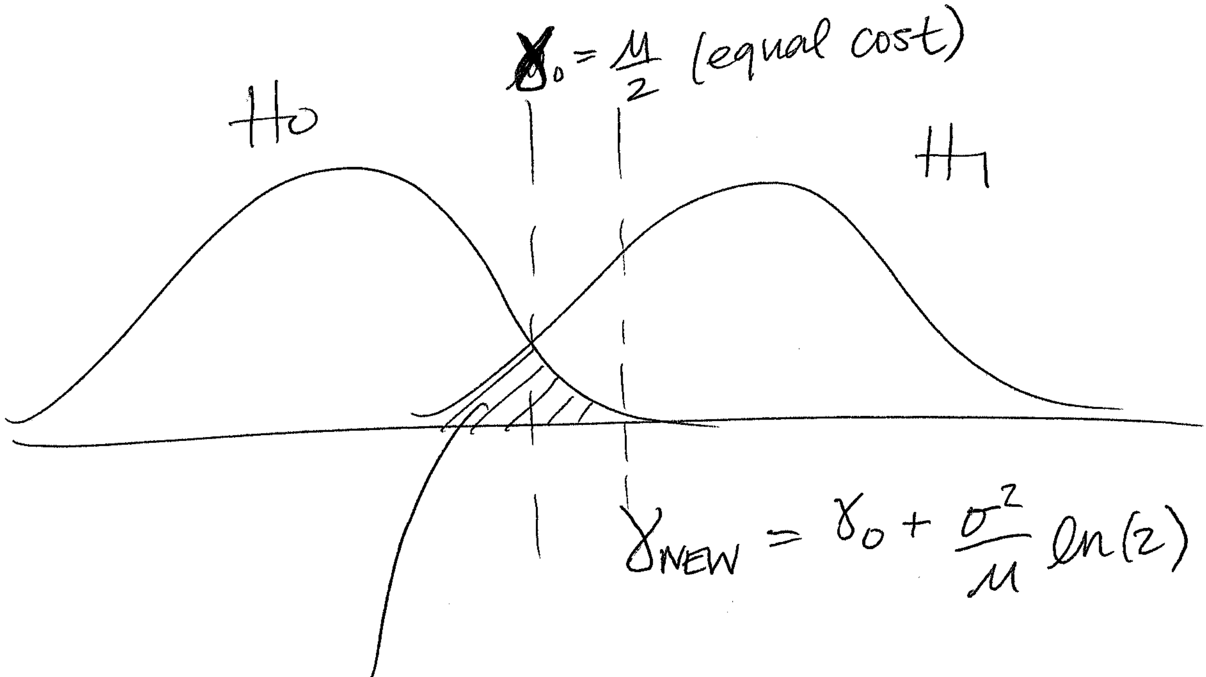
negative (shift to left)



$$\gamma = \gamma_0 + \frac{\sigma^2}{\mu}\left(\ln\left(\tfrac{1}{2}\right)\right)$$

$$\Rightarrow \quad R \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\mu}{2} + \frac{\sigma^2}{\mu}\left(\ln(2)\right) = \gamma$$

positive
(shifts γ to right)



WILL MORE OFTEN declare H_0 when H_1 is TRUE (MORE MISSES) but, less AREA ON OTHER SIDE, hence WE WILL HAVE <u>LESS</u> FALSE ALARMS ~ just as WE WANTED!

Finally,

if we have prior knowledge than one hypothesis is more likely than another, we want to take this into account

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Maximum-A-Posteriori (MAP) RULE

$P(H_0) = \pi_0, \quad P(H_1) = \pi_1$

looking @ posterior PROB (PROB. that H_i is true given the data)

$$P(H_1 | R) \underset{H_0}{\overset{H_1}{\gtrless}} P(H_0 | R)$$

Choose the hypothesis that is most likely given the data,

$$P(H_1|R) \underset{H_0}{\overset{H_1}{\gtrless}} P(H_0|R)$$

Using Bayes Rule,

$$\Rightarrow \frac{\pi_1 P(R|H_1)}{\cancel{P}(R)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0 P(R|H_0)}{\cancel{P}(R)}$$

$$\Rightarrow \boxed{\frac{P(R|H_1)}{P(R|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}} \quad \underline{\underline{\text{MAP}}}$$

if $\pi_0 = \pi_1 = \frac{1}{2}$, the MAP RULE IS the same AS Likelihood RATIO!