ELEC 303 Review Session #7

I. Transforms

$$M_x(s) = E[e^{sX}]$$

<u>example(1)</u>

Continuous X. ($N(0,1)$)

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty \le x \le \infty$$

$$M_X(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{sx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 - 2sx)/2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2 + sx - s^2/2} \cdot e^{s^2/2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{s^2/2} \underbrace{\int_{-\infty}^{\infty} e^{-(x-s)^2/2} \, dx}_{\text{valid gaussian pdf w/ mean } S,\ VAR = 1.}$$

normalization

$$= \frac{1}{\sqrt{2\pi}} e^{s^2/2} \cdot \sqrt{2\pi} = e^{s^2/2}$$

FOR A general gaussian w/ mean μ, VARIANCE σ^2, $Y \sim N(\mu, \sigma^2)$

$$Y = \sigma X + \mu.$$

$$M_Y(s) = E[e^{s(\sigma X + \mu)}] = e^{s\mu} E[e^{s\sigma X}]$$
$$= e^{s\mu} \cdot M(s\sigma).$$

$$\Rightarrow M_Y(s) = e^{s\mu} \cdot e^{(s\sigma)^2/2} = \underline{e^{(s^2\sigma^2 + 2s\mu)/2}}.$$

example (2) given $M_X(s)$, find $p_X(x)$.

$$M_X(s) = \frac{12-8s}{s^2-4s+3} = \frac{A}{s-3} + \frac{B}{s-1}.$$

$$A = (s-3)\cdot M_X(s)\Big|_{s=3} = \frac{12-8s}{s-1}\Big|_{s=3}$$

$$= \frac{12-8(3)}{2} = \frac{-12}{2} = \underline{-6}.$$

$$B = (s-1)\,M_X(s)\Big|_{s=1} = \frac{12-8(1)}{(1)-3} = \frac{4}{-2}$$

$$= \underline{-2}.$$

$$M_X(s) = \frac{-6}{s-3} + \frac{-2}{s-1} = \frac{1}{2}\Big(\underbrace{\frac{3}{3-s}}_{\text{transform of exp(3)}} + \frac{1}{1-s}\Big)$$

$$\Rightarrow \boxed{p_X(x) = \frac{1}{2}\left(3e^{-3x} + e^{-x}\right) \quad x \geq 0.}$$

Moment generating function

$$M(s) = \int_{-\infty}^{\infty} e^{sx} p_x(x)\, dx$$

$$\frac{d}{ds} M(s) = \int_{-\infty}^{\infty} \frac{d}{ds}(e^{sx})\, p_x(x)\, dx$$

$$= \int_{-\infty}^{\infty} x e^{sx}\, p_x(x)\, dx$$

When $s=0$

$$\frac{d}{ds} M(s)\Big|_{s=0} = \int_{-\infty}^{\infty} x\, p_x(x)\, dx = E[X]$$

$$\underbrace{\frac{d^2}{ds^2} M(s)\Big|_{s=0}}_{\text{moment-generating fn}} = \int_{-\infty}^{\infty} x^2\, p_x(x)\, dx = E[X^2]$$

moment-generating fn: To compute the n^{th} moment

$$E[X^n] = \frac{d^n}{ds^n} M(s)\Big|_{s=0}.$$

Central Limit Theorem (CLT)

Sample mean:

$M_n = (X_1 + \cdots + X_n)/n$

The weak law of large #'s states that the sample mean M_n is concentrated near the true mean (expectation) μ as $n \to \infty$.

<u>Sum</u>

$S_n = X_1 + \cdots + X_n = n \cdot M_n \longrightarrow \infty$

as $n \to \infty$.

Instead, consider the deviation of S_n from μ,

$$\boxed{Y = S_n - n\mu / \sigma\sqrt{n}} .$$

CLT says that the distribution of Y approaches A <u>normal</u> distrib.

More precisely,

Let $X_1, \ldots, X_n$ be a sequence of i.i.d. (independent identically distributed) RVS w/ mean μ, VARIANCE σ^2.

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}.$$

$$E[Z_n] = \frac{E[X_1 + \cdots + X_n] - n\mu}{\sigma\sqrt{n}} = 0.$$

and $$\mathrm{VAR}[Z_n] = \frac{\mathrm{var}(X_1) + \mathrm{VAR}(X_2) + \cdots + \mathrm{VAR}(X_n)}{\sigma^2 n}$$

$$= 1.$$

$\Rightarrow$ (CDF of $Z_n \to$ normal CDF)

$$\Rightarrow \quad \lim_{n\to\infty} P(Z_n \le z) = \underbrace{\Phi(z)}_{\text{CDF of A NORMAL dist.}} \quad \text{for all } z.$$

Normal Approximation based on CLT

Let $S_n = X_1 + \cdots + X_n$.

If n is large, $P(S_n \le c)$ can be approximated according to the following procedure.

1. Calculate mean $\underline{n\mu}$ and variance $\underline{n\sigma^2}$ of S_n.

2. Calculate $z = \dfrac{c - n\mu}{\sigma\sqrt{n}}$.

3. use appx:

$$P(S_n \le c) \approx \underbrace{\Phi(z)}_{\text{can find w/ STANDARD normal CDF table.}}$$

MIDTERM Review

(3) TRANSmitting bits over a noisy channel.

$$\begin{tikzcd} X \arrow[r] & \oplus \arrow[r] & Y \\ & N \arrow[u] & \end{tikzcd}$$

where
$$P_X(x) = \begin{cases} p, & x=0 \\ 1-p, & x=1 \end{cases}, \quad \underline{N \sim \mathcal{N}(\mu, \sigma^2)}$$

At the detector,

$$Y = X + N.$$

(a) $P_{Y|X}(y|x) = \ ?$

given x, the only <u>RANDOMNESS</u> in the signal is <u>N</u> $\Rightarrow$ given that $X = x_1$,

$$Y = N + \underset{\substack{\uparrow \\ \text{CONSTANT!}}}{x_1} \quad \Rightarrow \quad \underline{Y|X \sim \mathcal{N}(\mu + x_1, \sigma^2)}.$$

(b) $E[Y|X] = ?$

not random!!

$$E[Y \mid X = x_1] = E[N + x_1]$$

$$= E[N] + x_1$$

$$= \underline{\mu + x_1}.$$

(c) $E[Y] = ?$

<u>X,N are independ!</u>

$$E[X+N] = E[X] + E[N]$$

$$= (1-p) + \mu.$$

(d) <u>ERROR</u>

send A 1, $Y \le c$.

send A 0, $Y > c$.

$$P(\text{error}) = P(X=1) \cdot P[Y \le c \mid X=1]$$
$$+ P(X=0) \cdot P[Y > c \mid X=0]$$

$$P(\text{error}) = (1-p)\, P[N+1 \le c] + p \cdot P[N > c]$$

$$= (1-p)\cdot \underbrace{P[N \le c-1]}_{\Phi(c-1)} + p(1-\underbrace{P[N \le c]}_{\Phi(c)})$$

$$= (1-p)\,\Phi(c-1) + p\,(1-\Phi(c)).$$

What about $X \sim U(0,1)$?

(a) $P_{Y|X}(y|x) = ?$ (SAME)

(b) $E[Y|X] = ?$ (SAME)

(c) $E[Y] = ?$ $E[X] + E[N]$ (just a diff. expect. for X.)

How does $P_Y(y)$ differ
for $Y_1 = X_1 + N$, $Y_2 = X_2 + N$

where $X_1 \sim$ bernoulli (p).

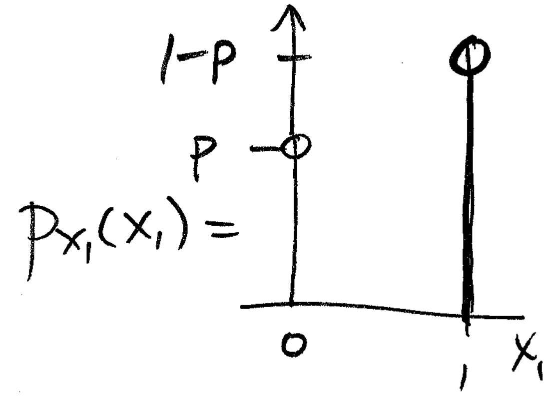
$X_2 \sim U(0,1)$.

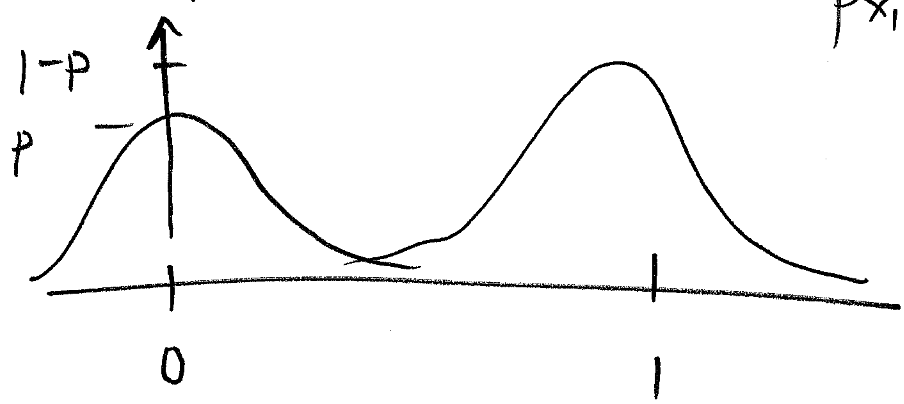
$N \sim N(\mu, \sigma^2)$.

For independent X, N :

(Convolution) !!

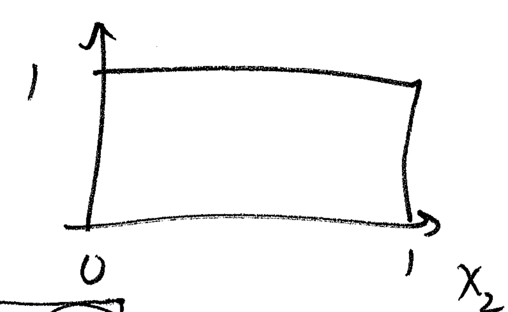
$P_{Y_1}(y) = P_{X_1}(x) \circledast P_N(n)$

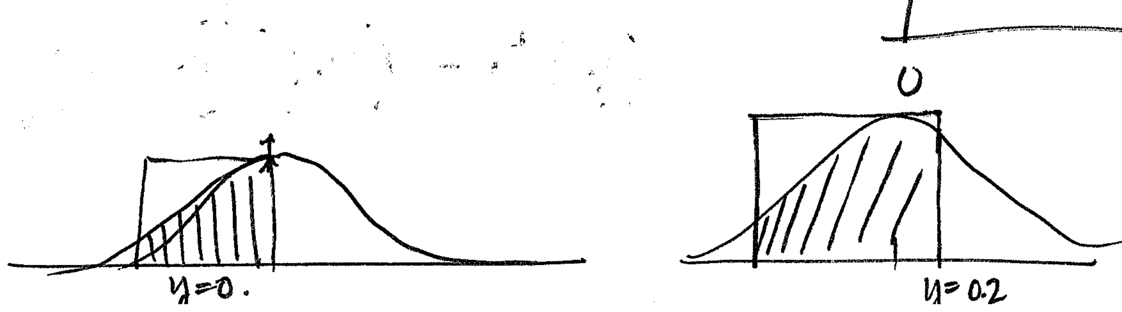




$P_{Y_2}(y) = P_{X_2}(x) \circledast P_N(n)$ $P_{X_2}(x)$

for a shift y





$$P_{Y_2}(y) = \underbrace{\int_{y-1}^{y+0} e^{-(n-\mu)^2/2\sigma^2}\, dn}_{\text{gaussian CDF} \hookrightarrow F_N(\cdot)}$$

$$\underbrace{\Phi(c) = \int_{-\infty}^{c} e^{-n^2/2}\, dn} = \text{CDF for MEAN}=0,\ \text{VAR}=\sigma^2.$$

$$\boxed{Z = \sigma \underbrace{N}_{\hookrightarrow N(0,1)\ (\text{STANDARD NORMAL})} + \mu.}$$

$$P_{Y_2}(y) = F_N(y) - F_N(y-1)$$

$$= \underline{\Phi\left(\frac{y-\mu}{\sigma}\right) - \Phi\left(\frac{y-1-\mu}{\sigma}\right)}.$$

For $\mathcal{U}(a,b)$:

$$* \quad P_Y(y) = \frac{1}{b-a}\left[\Phi\left(\frac{y+b-\mu}{\sigma}\right) - \Phi\left(\frac{y-a-\mu}{\sigma}\right)\right]$$