On the Impact of Finite Receiver Resolution in Fading Channels

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27 September 2006
Allerton Conference
Effects of Quantization

Quantization is becoming an issue in today’s high performance systems

Conflict of Interest

- New systems are demanding high bandwidth, but also high precision
- These goals can be antagonistic

Question: Can we measure how quantization affects our systems?

Our Contributions

- BER expressions for quantized, uncoded SIMO systems
- Error floors in BER
- Information-theoretic results for coded systems
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Analytical Framework

Need a useful performance metric:

- Bit error rates: difficult to compute
- Xia and Wang (2004): statistical BER expressions for SIMO, M-QAM systems

System Model

\[ r(n) = Hx(n) + N(n) \]
\[ r^q(n) = Q[Hx(n) + N(n)] \]

CSIR is assumed, though there is no automatic gain control at the receiver.

We will account for quantization in this model.
Modeling Quantization

Sheppard’s Correction

Quantization is a highly nonlinear process.

Uniform Quantization Model

\[ X^q = Q[X] = X + U \]
\[ U \sim \text{Unif}(q) \]

This uniform noise model allows for a moment update: Sheppard’s Correction

Sheppard’s Correction (1898)

\[ \text{var}[X^q] = \text{var}[X] + \frac{1}{12} q^2 \]

This correction relies on several important assumptions, can it be useful for us?
Modeling Quantization

Applying this result

We will apply this model to moment-based expressions for BER

**Probability of Error**

\[
pe = \frac{4}{M} \sum_{s_Q=1-\sqrt{M}}^{\sqrt{M}} \sum_{s_I=1-\sqrt{M}}^{\sqrt{M}} p_{s_Q,s_I}(R_{r_l r_l}, R_{r_l \hat{r}_l}, R_{\hat{r}_l \hat{r}_l})
\]

This is a functional expression, dependent only on three signal correlations \(E[r_l r_l], E[r_l \hat{r}_l], E[\hat{r}_l \hat{r}_l]\).

The variance terms \(E[r_l r_l]\) are updated with Sheppard’s Correction under certain conditions:

**Conditions Required for Update**

- pdf of \(r\) is “bandlimited:” characteristic function decays to zero
- Bin width \(q \leq q_{max}\)
- perfect CSIR (***)

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Modeling Quantization

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**Probability of Error**

\[ p_e = \frac{4}{M} \sum_{s_Q=1-\sqrt{M}}^{\sqrt{M}} \sum_{s_I=1-\sqrt{M}}^{\sqrt{M}} p_{s_Q,s_I}(R_{rlrl}, R_{rl\hat{r}l}, R_{\hat{r}l\hat{r}l}) \]

This is a functional expression, dependent only on three signal correlations \( E[r_{rl}] \), \( E[r_{rl}\hat{r}l] \), \( E[\hat{r}l\hat{r}l] \).

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Quantization Correction Theorem

Our Main Result

Given a uniformly quantized M-QAM SIMO system over a Rayleigh fading AWGN channel, received signal statistics are written as

\[ R_{r_l r_l}^{q} | n, s_Q, s_I = E[r_l r_l | n, s_Q, s_I] + \frac{1}{12} q^2 \]

\[ R_{\hat{r}_l \hat{r}_l}^{q} | n, s_Q, s_I = E[\hat{r}_l \hat{r}_l | n, s_Q, s_I] + \frac{1}{12} q^2 \]

\[ R_{\hat{r}_l r_l}^{q} | n, s_Q, s_I = E[\hat{r}_l r_l | n, s_Q, s_I] + \frac{1}{12} q^2 \]

for \( q \leq q_{max} \), where

\[ q_{max} = \frac{5}{3} \sqrt{1 + \sigma_N^2} \]

provided that

\[ \Phi_r(u) \to 0, \quad |u| > \frac{1}{q_{max}} \]
Consequences

Consequences of the update:

- The additive term bounds variances away from zero
- Even at high SNR, zero BER is not achievable

We can interpret this in the space of all possible channels: \( h \) Space

**Channel Space**

For \( |h| < q \), the channel is quantized to zero, and BER goes to 50%: full outage. This outage region is necessarily larger in a quantized system.
Results

Modulation Order and Quantization: 16 and 64 QAM, 4 Antennas

- **SNR**
- **BER**

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Results

BER as a function of SNR and Quantization, 16 QAM

- Full
- 8 bits
- 12 bits

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Coded System

Assumptions

Move to a more general system...

Model: \( Y = Q[hX + N] \)

Assumptions have been changed after the previous analysis...

System Configuration

- SISO Link
- Fading coefficient is distributed \( N(0, 1) \)
- SNR is adjusted by scaling noise: power constraint on \( X \)
- Uniform quantizer
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Achievable Rate of Quantized Channels

\[ R_{ach} = \max_{p(x)} \sum_{y} \int_{X} p(y|x)p(x) \log \frac{p(y|x)}{p(x)} \, dx \]

The nonlinearity of quantization makes this expression difficult to solve analytically.

Outage probability bounds frame error rate

Outage Probability

\[ P_{out}(R) = P[I(X;Y) < R] \]

Can be evaluated numerically.
Coded Systems: Information Theoretic Formulation

Classical Measures

Achievable Rate of Quantized Channels

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Coded Systems

Numerical Results

\[ p(x) \text{ for SNR = 17 dB, 16 mid-rise levels} \]
Coded Systems
Numerical Results

Blahut densities under power constraint, AWGN channel with 16 mid-rise levels
Coded Systems

Numerical Results

Probability of Outage: $R = 1.5$ bits

![Graph showing probability of outage vs. SNR]

Probability of Outage: $R = 1.5$ bits

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Continuing Work

Our Contributions

- Statistical update to BER expressions accounting for quantization in uncoded systems
- Demonstration of quantization-induced error floors in BER
- Error floors in outage plots: $d = 0$?

These results underscore the need for accurate Automatic Gain Control

There is still much to do!

- Incorporate Automatic Gain Control
- Derive approximate analytical solutions for achievable rates / outages
- Generalize to MIMO systems
- Examine diversity gain, multiplexing tradeoff

Thank you!!

Any Questions??
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