Bounds on Analog-Digital Quantization Error Effects on Multicarrier Symbols

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Quantization

Problem Formulation

All digital systems require quantizers!

Low-distortion quantizers with high precision and high bandwidth pose two problems:

- Cost
- Data overload
Quantization
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Effects on OFDM
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BER = 0.34!
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BER = 0.34!

How do we balance this tradeoff?

Effects on OFDM
Result
Resolution for Zero Error

Theorem: Minimum Required Resolution

\[ \bar{q} \leq \frac{2\pi \sqrt{2}}{\bar{d}N(\sqrt{M} - 1)} \]

1. Sketch of derivation
2. Interpretation
3. Simulation Results
4. Example
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System Model
Canonical Quantization System

\[
d[k] \rightarrow \text{IDFT} \rightarrow \text{DFT} \rightarrow \text{Detect}
\]

\[
\begin{align*}
\text{Re}[s[n]] & \\
\text{Im}[s[n]] & \\
\end{align*}
\]

\[
r[n] = Q(\text{Re}[s[n]]) + jQ(\text{Im}[s[n]])
\]

Parameters
\[
\bar{q} - \text{Bin width} \\
N - \text{Subcarriers} \\
M - \text{Modulation order}
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System Model
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Parameters
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The Quantization Object

Functional Decomposition

\[ \bar{q} = 1 \]

\[ Q(x) \]

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The Quantization Object

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The Quantization Object

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**The Quantization Object**

**Functional Decomposition**

\[
Q(x) = x + S(x) = x + \sum_{l=1}^{\infty} \frac{\bar{q}}{l\pi} \sin \left( \frac{2\pi x l}{\bar{q}} \right)
\]

\(\bar{q} = 1\)

---

**Graphical Representation**

The graph shows the functions \(Q(x)\), \(S(x)\), and \(Q(x) - S(x)\) for different values of \(x\). The quantization process is illustrated with steps and the sum formula is applied to demonstrate the decomposition of the quantization function. The graph is labeled with axes and values, highlighting the quantization steps and the sinusoidal nature of the decomposed function.
Analysis
Finding Quantization Error Terms

\[ r[n] = s[n] + \bar{q} \sum_{l=1}^{\infty} \frac{1}{l\pi} \sin \left( \frac{2\pi s_i[n]l}{\bar{q}} \right) + j\bar{q} \sum_{l=1}^{\infty} \frac{1}{l\pi} \sin \left( \frac{2\pi s_q[n]l}{\bar{q}} \right) \]
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\[ r = \underbrace{s}_{\text{Original}} + \underbrace{S_id + j \cdot S_qd}_{\text{Quantization}} \]
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\[ r = \begin{cases} s & \text{Original} \\ S_i d + j \cdot S_q d & \text{Quantization} \end{cases} \]

\[ \| (S_i + j \cdot S_q) d \|_2 \]
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\[ \| (S_i + j \cdot S_q) d \|_2 \leq E_{\bar{q}} \equiv \bar{q} \bar{d} \sqrt{N} \]
\[ \pi \sqrt{2} \]
Parseval’s Theorem gives error energy in the symbol domain:

\[ E^F_{\bar{q}} = \frac{\bar{q}dN}{\pi \sqrt{2}} \]

Final Result

\[ \bar{q} \leq \frac{2\pi \sqrt{2}}{dN(\sqrt{M} - 1)} \]
Interpretation

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Resolution moves…

- inversely to the subcarriers \( N \)
- inversely to the root of modulation order \( M \)

Intuition

- Error is “spread” over subcarriers, but all subcarriers must still be resolveable at the receiver.
- Increasing modulation order, while maintaining constant power, requires increased resolution in the symbol domain.
- Other constants fall from the definition of the transform.
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\[ \bar{q} \leq \frac{2\pi \sqrt{2}}{dN(\sqrt{M} - 1)} \]

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Simulation Results

Total Error Energy

Actual error energy is only slightly lower than predicted.
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Symbol Error

Symbol Error Energy
Linear in $\bar{q}$

$\bar{q} = 0.1$
$E_F^{\bar{q}} = 0.3272$

$\bar{q} = 0.01$
$E_F^{\bar{q}} = 0.0329$
Simulation Results

Symbol Error

Symbol Error Energy

Linear in $\bar{q}$
Simulation Results
System Performance

Our theorem guarantees zero error!
Simulation Results

System Performance

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Numerical Example
IEEE 802.16e : WiMAX

General system parameters:
- 25 MHz bandwidth
- 2 channels (I and Q)
- 64, 256, 1024 carrier OFDM
- 16 QAM

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\begin{array}{|c|}
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N = 256 \\
14 \text{ bit ADC} \Rightarrow 700 \text{ Mbps} \\
Theorem requires 11 bits \\
\Rightarrow 550 \text{ Mbps}, > 20\% \text{ savings} \\
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Continuing Work

Where to now?

Contribution

- A measure of quantization error energy for multicarrier systems
- A bound on resolution required for zero error

Open Issues

- “Smearing” of error across subcarriers
- More realistic channels
- Effects on channel estimation
- etc. . .
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Thank you!
Any questions?