Non-Type-wise Method for Sharper Upper Bounds

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In this note, we outline an idea that one can use to tighten bounds obtained using the method of types. This is based on problems 7, 11 and 12 from Chapter 2 of Csiszar and Korner’s book.

A brief note about notation before we proceed. \( \mathcal{X} \) denotes a finite set and let \( x \in \mathcal{X}^k \) denotes the length \( k \) vector \( (x_1, x_2, \ldots, x_k) \) from \( \mathcal{X}^k \). Given a bounded function \( M \) on \( \mathcal{X} \), \( M^k(x) := \prod_{i=1}^{k} M(x_i) \) and for any \( F \subset \mathcal{X}^k \), \( M^k(F) := \sum_{x \in F} M^k(x) \). \( \mathbb{1}\{\cdot\} \) denotes the indicator function. Further, we use standard notation for information theoretic quantities.

The main result can be stated as follows.

**Theorem 1.** Let us suppose that we have a bounded “mass” function \( M(\cdot) \) on \( \mathcal{X} \). Then, for any \( F \in \mathcal{X}^k \), the following holds

\[
M^k(F) \leq \exp\{-kD(P_{M,F} \parallel M)\}
\]

where \( P_{M,F}(a) := \sum_{x \in F} \frac{M^k(x)}{M^k(F)} \mathbb{P}_x(a) \), for all \( a \in \mathcal{X} \) and \( D(P_{M,F} \parallel M) := \mathbb{E}_{P_{M,F}} \left[ \log \left( \frac{P_{M,F}}{P_M} \right) \right] \).

**Proof.** To prove this, we define a random vector \( X^k = (X_1, X_2, \ldots, X_k) \) which is drawn according to the following distribution

\[
\Pr\{X^k = x\} := \frac{M^k(x)}{M^k(F)} \mathbb{1}\{x \in F\}
\]

and independently of \( J \sim \{1, 2, \ldots, k\} \). First, we upper bound the joint entropy of \( X^k \) by the sum of individual entropies and proceed as follows

\[
H(X_1, X_2, \ldots, X_k) \leq \sum_{i=1}^{k} H(X_i)
\]

\[
= k \sum_{i=1}^{k} \frac{1}{k} H(X_i)
\]

\[
= h(X_j|J)
\]

\[
\leq h(X_j)
\]

Now observe that \( P(X_j = a) = \sum_{i=1}^{k} \frac{1}{k} \sum_{x} P(X^k = x) = P_{M,F}(a) \). This gives us the following upper bound

\[
H(X_1, \ldots, X_k) \leq -k \sum_{a \in \mathcal{X}} P_{M,F}(a) \log P_{M,F}(a)
\]
Alternatively, the above joint entropy can be evaluated as
\[ H(X_1, X_2, \ldots, X_k) = -\sum_{x \in F} \frac{M^k(x)}{M^k(F)} \log \left( \frac{M^k(x)}{M^k(F)} \right) \] (4)
\[ = \log M^k(F) - \sum_{x \in F} \frac{M^k(x)}{M^k(F)} \log M^k(x) \] (5)
\[ = \log M^k(F) - \sum_{x \in F} \frac{M^k(x)}{M^k(F)} k \sum_{a \in X} P_x(a) \log M(a) \] (6)
\[ = \log M^k(F) - k \sum_{a \in X} P_{M,F}(a) \log M(a) \] (7)

(3) and (7) together conclude the proof.

**Corollary 1.** For any \( F \in X^k \), the following bound on the size of \( F \) holds
\[ |F| \leq \exp \{ kH(P_{1,F}) \} \] (8)
where \( P_{1,F}(a) := \frac{1}{|F|} \sum_{x \in F} P_x(a) \). Further, for any distribution \( Q \) on \( X \), we have
\[ Q^k(F) \leq \exp \{ -kD(P_{Q,F} \| Q) \} \] (9)
where \( P_{Q,F}(a) := \sum_{x \in F} \frac{Q^k(x)}{Q^k(F)} P_x(a) \) for all \( a \in X \).

**Proof.** For the first part, set \( M(a) = 1 \) for all \( a \in X \) and the second part follows by setting \( M(a) = Q(a) \) for all \( a \in X \).

This corollary tells us that the size of an arbitrary set in \( X^k \) can be bounded in terms of the entropy of the “average type of the sequences of that set” and that a similar statement holds for \( Q^k(F) \).

To see why these results are useful, we now consider three examples. In each of these cases, standard type-based arguments would give us exponential bounds which are tight only as \( k \to \infty \). But, using the “non-typewise” bounding of Theorem 1, we show that under some conditions, the same bounds hold non-asymptotically.

**1. Error Exponent for Binary Block Codes.** We now show that for any finite set \( X \) and rate \( R > 0 \), there exists a \( k \)-to-\(-n_k \) block code such that for any DMS with alphabet \( X \) and arbitrary distribution \( P \), the probability of error satisfies
\[ P_e \leq \exp \left\{ -k \min_{Q: H(Q) \geq R} D(Q \| P) \right\} \] (10)

Observe that this bound does not involve a polynomial factor as is usual in proofs by the method of types. To see this, let \( A_k := \bigcup_{Q: H(Q) < R} T_Q \). The encoding function essentially maps one-to-one from \( A_k \) to an integer from \( \{1, 2, \ldots, 2^k R\} \) and anything in \( X^k \setminus A_k \) is mapped to 1 (say). By defining \( Q(\cdot) = \sum_{x \in A_k} \frac{P^k(x)}{P(X^k)} P_x(\cdot) \), we can use the results of Corollary 1 to get
\[ P \left( X^k \setminus A_k \right) = P \left( \left\{ x \in X^k : H(P_x) \geq R \right\} \right) \] (11)
\[ \leq \exp \left\{ -k D(Q \| P) \right\} \] (12)

(10) follows directly from this since, by the concavity of entropy, we know that \( H(Q) \geq R \).
2. **Sanov’s Theorem.** Let $\mathcal{P}$ be a set of distributions on the alphabet $\mathcal{X}$ and let $Q$ be another distribution on $\mathcal{X}$. Sanov’s theorem gives us a bound on the probability that a random sample drawn according to $Q$ would “appear as though it was drawn from a distribution in $\mathcal{P}$”. This bound is $(k + 1)^{|\mathcal{X}|} \exp\{-k \inf_{P \in \mathcal{P}} D(P \| Q)\}$. However, if $\mathcal{P}$ is a convex set of distributions, then the following tighter bound holds

$$\frac{1}{k} \log Q^k \left( \left\{ x \in \mathcal{X}^k : P_x \in \mathcal{P} \right\} \right) \leq - \inf_{P \in \mathcal{P}} D(P \| Q)$$  \hspace{1cm} (13)

and this also follows as a direct consequence of Corollary 1.

3. **Hypothesis Testing.** Following along the same lines, we can get a stronger result for the probability of missed detection in hypothesis testing. We can actually show the following statement:

For any given $P$ and $a > 0$, there exists $A_k \subset \mathcal{X}^k$ such that

$$\lim_{k \to \infty} \frac{1}{k} \left( 1 - P^k(A_k) \right) = -a$$ \hspace{1cm} (14)

and for every $Q$

$$\frac{1}{k} \log Q^k(A_k) \leq - \min_{\hat{P} : D(\hat{P} \| P) \leq a} D(\hat{P} \| P)$$ \hspace{1cm} (15)