Chapter 2

Mechanics

From the mechanical point of view, the goal of the electromechanical design process outlined in Chapter 1 is to cause an object or objects (the *load*) to perform a specific, controlled *motion*. In this chapter we will examine how the load, the mechanism, and the electrically produced forces interact to produce motion.

2.1 Motion

2.1.1 Examples of Motion

Let's examine some of the items from the product gallery in Chapter 1 and see what kinds of motion we might need to produce.

Drill. In the economy model drill, the chuck turns at about 1000 rpm when the trigger is pulled, and will slow down when drilling, the actual speed depending on the size of the drill and the toughness of the material being drilled. As we pay more money, we get additional features such as selectable speed, constant speed under load, reversible direction, and a torque limiter for driving screws.

Elevator. Here we wish to control a position rather than a velocity. More specifically we want to be able to select one of a fixed set of vertical positions for the car. Although our goal is the steady state position following a move, we must also control the transitions between stops (i.e. the velocity and acceleration) so that the ride is safe and comfortable.

CD Player. We could think of the desired action here as a single motion: moving the focal point of the lens along the track at a fixed tangential velocity. Or we could look at it as three coordinated motions: spinning the disk, following the track, and staying in focus. In any case, the motion is produced by three seperate actuators.

Robot. So far, the motions we've seen have either been one dimensional or have been a set of independently controlled one dimensional motions. In a robot, although made up of a combination of single dimensional motions, the desired end product is a completely arbitrary six degree of freedom (6 DOF) motion. This flexibility allows a robot to assume any position and any orientation (within its work envelope) and to move between them along any path. Depending on the application, we may require accuracy only of the end points, or of all points along the path.

Furby. The Furby has a large vocabulary of motions: opening and closing his eyes and mouth, rocking back and forth, and wiggling his ears. These are combined in a number of patterns depending on whether he is eating, sleeping, dancing, or chattering. It is significant that each of the individual motions is cyclic.

Washing Machine. The typical top loading washing machine has four moving parts: the tub, the agitator, the pump, and the water valve. During the wash and rinse cycles, the tub is stationary and the agitator oscillates back and forth. At the beginning of these cycles, the water valve opens until the tub is filled, then closes. During the spin cycles, both the tub and agitator rotate at high speed, and the pump runs to empty the water from the tub.

2.1.2 Types of Motion

From these examples and others, we can collect a number of qualitative characterizations about the types of motion a system might be required to produce. These are summarized in Table 2.1

Regulation of fixed steady state value	
	Washing machine spin cycle
	Automobile cruise control
Tracking a slowly varying value	
	CD focus
	Heliostat (solar collector)
Continuously adjustable steady state value	
	Drill
	CD rotation
Multiple discrete fixed values	
	Washing machine water valve
	Elevator
Following an arbitrary *path*	
	CD track following
	Numerically controlled lathe
Multiple coordinated motions	
	Furby
	Assembly robot

Table 2.1: Types of Motion

We can describe these motions by saying that we want to be able to set the value of $m(t)$ where m is a mechanical quantity (position or velocity) and t may be a specific value or set of values, or in the case of steady state, $t = \infty$.

2.1.3 Characteristics of Motion

Here is a catalog of some of the quantitative characteristics of motion that we will need to be able to achieve.

The Controlled Quantity The *primary* physical variable being controlled, e.g. the vertical position in the elevator or the rotational velocity in the drill. In a linear system this may be a position, velocity, or force, or in a rotational system an angular position, angular velocity, or torque.

Number of Desired States This may be a set of discrete values (elevator, water valve) or a continuum (robot, CD player).

Range The motion may be required over a limited range, or may be unlimited. In the washing machine, the agitator rotates over a $\pm 90°$ range in the wash and rinse cycles, while in the spin cycle the agitator and tub rotate over an unlimited range.

Load If the desired quantity is a position or velocity, how much force will be required to sustain it?

Accuracy How closely must the actual value match the desired value?

Constraints on Path Must the actual value be within the accuracy requirements at all times (CD player tracking) or only at the endpoints (elevator)? Are there requirements for "smoothness" of motion such as maximum values for velocity, acceleration, or jerk?

2.1.4 Mechanisms

Responsibility for producing the desired motion is shared between the mechanism and the controller. In many cases the *geometry* of the path is constrained by the mechanism (often to one dimension) and the *traversal* of the path is directed by the controller.

Some components of the mechanism are responsible for implementing constraints on the motion (for example, the rails in the elevator or the spindle bearing in the drill), others provide support for the load (the elevator cage or the drill chuck), some transform motion (windlass, gearbox) or move it from one place to another (cable, shafts), and some are responsible for holding everything else together in the proper orientation.

We could regard the mechanism as a black box which attaches to the load on one side and to the electromechanical devices on the other. The motion presented at the electromechanical side of this box has usually been decomposed and transformed into a set of single dimensional linear and rotational motions. For example, a robot arm takes an arbitrary 6 DOF motion and decomposes it into a set of six rotations which must be coordinated by the controller.

2.1.5 Electromechanical Devices

Having constrained and decomposed the desired motion down to a set of simple, one dimensional motions we can now animate our mechanism by attaching electrically controlled *actuators* to these "ports." Ideally, an actuator would convert an electrical signal (e.g. a voltage) in a proportional manner to the desired mechanical quantity (e.g. a displacement or a velocity). Similarly, to monitor motion a *sensor* should convert the mechanical quantity being measured to a corresponding electrical quantity. We will refer to actuators and sensors jointly as *electromechanical devices.*

2.2 Mechanical Forces

If we attempt to change the position or velocity of a part of our mechanism it responds with a force. These forces must be matched or overcome if we are to achieve the desired motion.

2.2.1 Linear Inertia

Newton's Laws Consider the simple mechanical system in Figure 2.1. According to

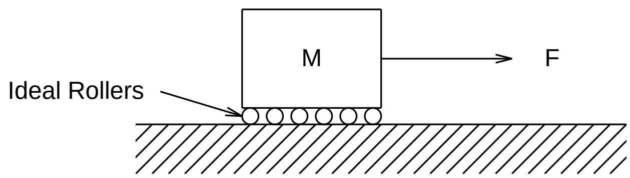

Figure 2.1: One-dimensional, Frictionless Mechanical System

Newton's First Law, the box will remain stationary in the absence of any applied forces. To move the box to a different position, it is necessary to apply appropriate forces.

Newton's Second Law tells us what the result of these forces will be:

$$F \propto ma$$

where F is the force acting on the object, m is its mass, and a is the resulting acceleration. If an appropriate system of units is chosen, the constant of proportionality becomes one. We will utilize SI (Système International) units for the most part, as they minimize the number of extraneous constants in various formulas. In this case, for F in Newtons, m in kilograms, and a in meters/second^2, we have the familiar:

$$F = ma \tag{2.1}$$

2.2.2 Rotational Inertia

Many of the devices we will be considering produce rotary, rather than straight line motion. Figure 2.2 is the rotational equivalent of Figure 2.1. If a tangential force is applied to a cylinder of radius r, supported by its axis, it will experience a *torque* of

$$T = rF$$

If it is free to rotate, the torque will produce an angular acceleration

$$T = J\ddot{\theta}$$

where J is the *moment of inertia* of the cylinder about its axis. We may also write

$$T = J\dot{\omega} \tag{2.2}$$

where ω is the *angular velocity* in radians/second.

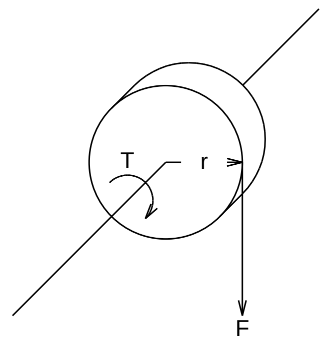

Figure 2.2: One-dimensional, Frictionless Rotational System

Moment of Inertia Moment of inertia is the rotational counterpart to mass, and represents the ease (or difficulty) with which a torque can accelerate an object. For a solid cylinder, the moment of inertia about its axis is

$$J = \frac{1}{2}mr^2$$

For a hollow cylinder with inner radius r_1 and outer radius r_2

$$J = \frac{1}{2}m({r_1}^2 + {r_2}^2)$$

2.2.3 Gravity

If Figure 2.1 was taken on Earth, there is always at least one force acting on the box. Newton's Law of Gravitation tells us that there is a force:

$$F = G\frac{m_{box}m_{earth}}{{r_{earth}}^2}$$

$$F = mg \tag{2.3}$$

acting vertically downward on the box. Newton's Third Law tells us that there is an equal reaction force from the table acting upward on the box. Hence the net force is zero and in the absence of any additional forces, the box remains stationary.

If a rotational system is statically balanced, gravitational forces will cancel and there will be no net gravitationally induced torque, regardless of orientation.

2.2.4 Friction

The idealized systems in Figures 2.1 and 2.2 will continue to move forever if undisturbed. Any real system will contain *friction* which will eventually arrest the motion.

Friction comes in two basic flavors. If we replace the rollers in Figure 2.1 with an "ideal" oil film, we have what is called *viscous friction* which causes a force which is proportional to the velocity and opposite the direction of motion.

$$F = -cu = -c\dot{x}$$

Note that we will use $u = \dot{x}$ to denote linear velocity to avoid confusion with the use of v to denote voltage.

The other common form of friction is *dry* or *Coulomb* friction. It is similar in that it acts in the direction opposite to the motion (or impending motion) of the objects, but the magnitude is proportional to the normal force between the two objects.

$$F = -\mu N$$

where μ is called the *coefficient of friction.* Section A.4.1 gives the approximate coefficient of friction for several common material pairs.

Coulomb friction is also dependent on velocity in the following sense: the value of μ just prior to the onset of motion, called the *coefficient of static* or *limiting friction*, is usually different from the value after motion has begun, called the *coefficient of sliding* or *kinetic friction.* Typically sliding friction is slightly less than static friction. This characteristic causes the phenomenon known as *stick-slip* which is responsible for the sound of the violin and squealing brakes, and can cause instability in control systems when executing small motions.

2.2.5 Elastic Force

Just as the bearings of a real mechanical system are not ideal (frictionless), neither are the structural elements perfectly rigid. If a force is applied to a member which is constrained at some other point it will deflect.

In most cases, for "reasonably" small forces, the amount of deflection is proportional to the amount of applied force. This phenomenon is known as *elastic deformation* and its linear regime is described by *Hooke's Law*:

$$F = -kx \tag{2.4}$$

where k is called the *spring stiffness.*

In a rotational system, a torsional spring produces the following relation between angular displacement and the resulting torque:

$$T = -k\theta$$

2.3 Work and Energy

Work is performed when a force is applied through a distance:

$$\Delta W = F \cdot \Delta x \tag{2.5}$$

for a constant force, or more generally:

$$\begin{aligned} dW &= F\, dx \\ W &= \int F\, dx \end{aligned} \tag{2.6}$$

Work done *on* a system increases the *energy* of the system. Conversely, work may be done *by* a system, thereby decreasing its energy.

Kinetic Energy In accelerating a mass m with a force F:

$$F = ma = m\dot{u}$$

$$\begin{aligned} W &= \int F\, dx \\ &= \int m\dot{u}\, dx = m \int u\dot{u}\, dt = \frac{1}{2} m u^2 \end{aligned} \tag{2.7}$$

For a rotating system:

$$W = \frac{1}{2} J \omega^2 \tag{2.8}$$

Strain Energy For a spring of stiffness k compressed through a distance x:

$$\begin{aligned} F &= kx \\ W &= \int F\, dx = \int kx\, dx = \frac{1}{2} k x^2 \end{aligned} \tag{2.9}$$

Gravitational Potential Energy In moving a mass m vertically in a gravitational field:

$$\begin{aligned} F &= mg \\ W &= \int F\, dx = \int mg dx = mgx \end{aligned} \tag{2.10}$$

It is important to remember that work entails both force and distance, and that arbitrarily large forces may be applied or arbitrarily large distances traversed without the expenditure of any energy, if the corresponding distances or forces are zero.

2.3.1 Conservation of Energy

Conservative Forces The energy that these forces (inertial, elastic, gravitational) impart to a system is stored in a form (kinetic, strain, gravitational potential) which is *recoverable*. If the process which put the energy into the system is reversed (by decelerating the mass, releasing the spring, or lowering the weight) the energy may be taken back without loss.

The forces involved in recoverable energy transfer are called *conservative forces*. Another definition of a conservative force is that the total work performed in traversing a closed cycle is zero.

The law of *conservation of energy* states that in a system involving conservative forces, the sum of all energies in the system is constant.

Non-conservative Forces The work expended in overcoming a frictional force may not be recovered by reversing the process. Forces such as friction are termed *non-conservative* or *dissipative forces*. Mechanical energy in a system involving such forces is not conserved but is changed into another form such a heat which is dissipated and is not recoverable.

Conservation of energy encompasses dissipative forces as well, if thermal energy is included. The total energy of a system remains constant, but that part of it involving dissipative forces is converted to heat (or some other form of energy) and is not directly recoverable.

2.4 Dynamics

2.4.1 Equations of Motion

2.4.1.1 Free Response

For the system in Figure 2.3 the forces acting on the mass are the restoring force due to the extension or compression of the spring, and the viscous drag force from the damper. We may write:

$$F = ma = m\ddot{x} = -kx - c\dot{x}$$

or

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.11}$$

If we define the *undamped natural frequency*

$$\omega_n = \sqrt{\frac{k}{m}} \tag{2.12}$$

the *damping coefficient*

$$\alpha = \frac{c}{2m} \tag{2.13}$$

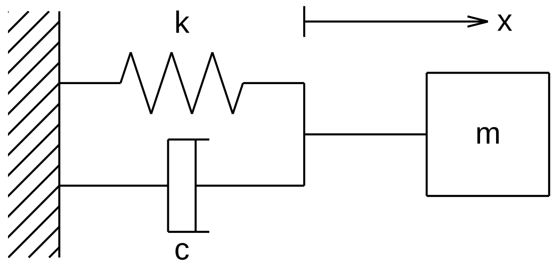

Figure 2.3: Second-order, Unforced Mechanical System

and the *damping ratio*

$$\zeta = \alpha/\omega_n \tag{2.14}$$

then solutions to this equation will have one of three forms:

Overdamped ($\zeta > 1$)

$$x = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where

$$s_1 = -\alpha + \sqrt{\alpha^2 - {\omega_n}^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - {\omega_n}^2}$$

Critically Damped ($\zeta = 1$)

$$x = (k_1 + k_2 t)e^{-\alpha t}$$

Underdamped ($\zeta < 1$)

$$x = ke^{-\alpha t}\cos(\omega_d t + \theta)$$

where

$$\omega_d = \sqrt{{\omega_n}^2 - \alpha^2}$$

is the *damped natural frequency*.

2.4.1.2 Forced Response

If an external force $F(t)$ is applied to the mass, as in Figure 2.4, (2.11) becomes

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{2.15}$$

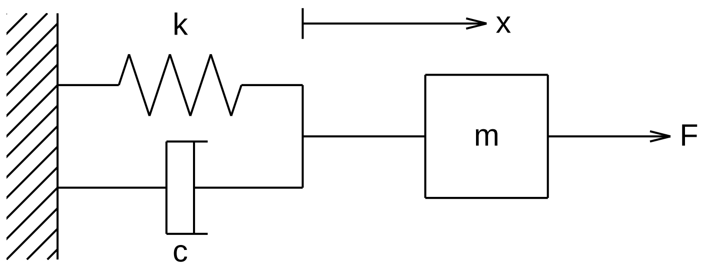

Figure 2.4: Second-order, Forced Mechanical System

The solution of (2.15) is of the form

$$x(t) = x_c(t) + x_p(t)$$

i.e. it consists of the sum of a transient solution, x_c (the *complementary function*), and a steady state solution, x_p (the *particular integral*). The complementary function is a solution of the homogeneous system (2.11). The particular integral depends on both the dynamics of the system and the forcing function. Two special cases for the applied force are of interest:

Step Response If $F(t) = 0$ for $t < 0$ and $F(t) = F_0$ for $t > 0$, then $x_p(t) = F_0/k$. Figure 2.5 shows a typical step response for each of the classes of damping.

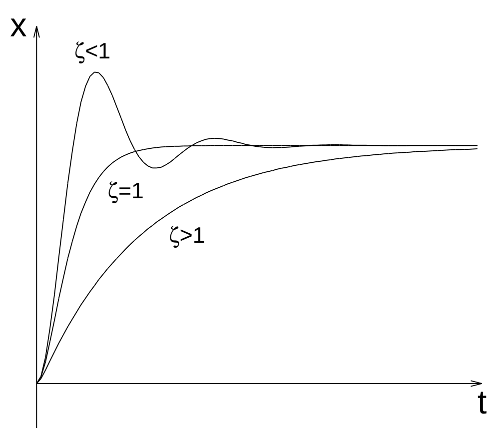

Figure 2.5: Step Response.

Sinusoidal Response When the excitation is sinusoidal, the steady state response is also sinusoidal at the same frequency, with amplitude depending on both the damping ratio and the ratio between the driving frequency and the undamped natural frequency. There is also a phase shift between the input and the response which varies from zero at low frequencies, through 90 deg at the undamped natural frequency, to 180 deg at high frequencies.

For an excitation of $F(t) = F_0 cos(\omega t)$, the steady state response is

$$x(t) = X cos(\omega t - \phi)$$

where

$$X = \frac{X_0}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$tan\phi = \frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

and

$$X_0 = F_0/k$$

Figure 2.6 shows the magnification ratio X/X_0 for various damping ratios.

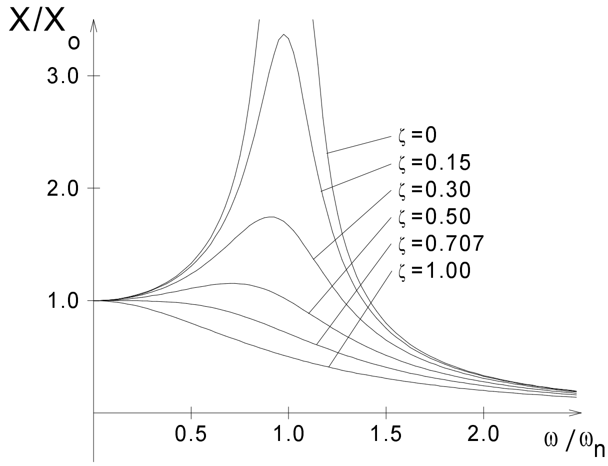

Figure 2.6: Frequency Response vs. Damping Ratio.

2.5 Control

As we'll see in Section 3.1, the only mechanical quantity which can be made to directly correspond to an electrical voltage or current is force (or torque). What if we want to control a position or velocity? We have a number of alternatives.

1. We will develop a number of actuators for which the *steady state* position or velocity is determined by a voltage or current (or frequency).

2. We can combine an electrically controlled force with an elastic element whose force varies with displacement or a viscous element whose force varies with velocity. In this case the equilibrium position or velocity will be determined by the electrical variable.

3. We can create a closed loop control system where the electrically produced force is applied in such a way as to produce motion which will reduce the difference between the desired position or velocity and its actual value.

2.5.1 Open Loop Control

Figure 2.7 shows how we might use method 2 to control the position of the elevator. Assume the motor produces a torque which is proportional to the applied current: $T = K_T i$. This will produce an upward force of $F_{up} = \frac{T}{r} = \frac{K_T}{r} i$ on the cage. If m is the mass of the cage, k the spring constant, and c is the viscous friction of the oil on the rails, then the total downward force is $F_{down} = m\ddot{x} + c\dot{x} + kx + mg$. Since these must be equal, we have

$$m\ddot{x} + c\dot{x} + kx + mg = \frac{K_T}{r} i \qquad (2.16)$$

In the steady state we get $kx + mg = \frac{K_T}{r} i$ which gives

$$x(i) = \frac{K_T}{rk} i - \frac{mg}{k}$$

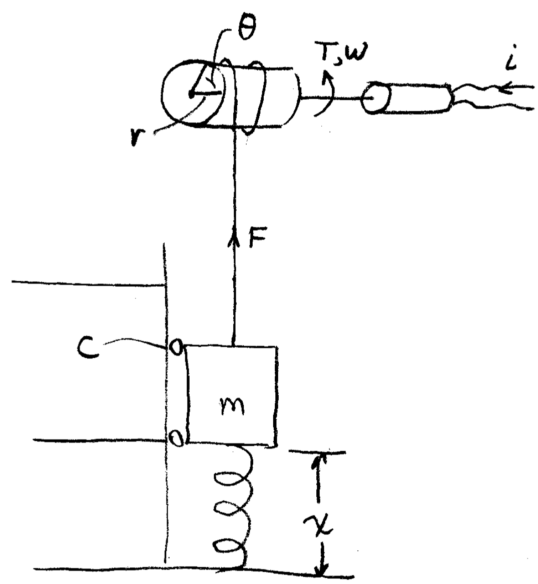

Figure 2.7: Open loop elevator control

Solving this for i tells us how to set the current to achieve a desired position

$$i(x) = \frac{r}{K_T}(kx + mg)$$

Note that while this current will determine the steady state position of the car, the actual position as a function of time when moving between floors will be given by the solution of 2.16. Also, this control law is only valid for an empty car. If any passengers get into the car, the total weight will increase and the car will sink.

2.5.2 Closed Loop Control

Suppose that we have a sensor whose output is proportional to the vertical position of the car (x) or alternately, to the angular position of the drum (θ) such that $v_x = K_x x$. Let $v_{des} = K_x x_{des}$ be a controlling signal which represents the desired position of the car (x_{des}). Then $v_{err} = v_{des} - v_x$ is an *error signal* which will be zero when the desired and actual positions are the same. If we amplify this signal and use it to drive the motor in the previous example, i.e. $i = G(v_{des} - v_x) = GK_x(x_{des} - x)$ then

$$m\ddot{x} + c\dot{x} + kx + mg = \frac{K_T}{r}A(x_{des} - x) \tag{2.17}$$

where $A = GK_x$. In the steady state we get $kx + mg = \frac{K_T A}{r}(x_{des} - x)$ which gives

$$x(v_{des}) = \frac{K_T A/r}{k + K_T A/r}x_{des} - \frac{mg}{k + K_T A/r}$$

Note that $\lim_{G\to\infty} x(v_{des}) = x_{des}$. This says that if we make the amplifier gain G sufficiently large the error caused by changing the load in the car can be made arbitrarily small. Also as we increase G the effect of the spring (k) becomes negligible and we can remove it.

However, these benefits don't come for free. Let's examine the dynamic behavior of the closed loop system.

$$\begin{aligned} m\ddot{x} + c\dot{x} + kx &= \frac{K_T A}{r}(x_{des} - x) - mg \\ m\ddot{x} + c\dot{x} + (k + \frac{K_T A}{r})x &= \frac{K_T A}{r}x_{des} - mg \end{aligned}$$

Define $w = mg$ to be the weight of the car and its contents and take the Laplace transform.

$$ms^2X(s) + csX(s) + (k + \frac{K_T A}{r})X(s) = \frac{K_T A}{r}X_{des}(s) - W(s)$$

Let $k_1 = k + \frac{K_T A}{r}$. We can define two transfer functions

$$H_c(s) = \frac{X(s)}{X_{des}(s)} = \frac{K_T A/r}{ms^2 + cs + k_1}$$

gives the response of the car position X to the controlling input X_{des} and

$$H_d(s) = \frac{X(s)}{W(s)} = -\frac{1}{ms^2 + cs + k_1}$$

represents the *disturbance* caused by variations in the weight of the car.

The natural response of the system is given by the roots of the characteristic equation

$$ms^2 + cs + k_1 = 0$$

which gives

$$\begin{aligned}
\omega_n &= \sqrt{\frac{k_1}{m}} \\
\alpha &= \frac{c}{2m} \\
\zeta &= \frac{\alpha}{\omega_n} = \frac{c/2m}{\sqrt{k_1/m}} = \frac{cm}{2\sqrt{k_1}}
\end{aligned}$$

As we increase A to improve the accuracy, the system will eventually become underdamped and oscillate with changes in position or load. To prevent this, we must increase the damping by increasing either c or m, neither of which is desirable. However, we can increase the *effective* damping by incorporating additional feedback. Note that adding *position* feedback increased the *stiffness* of the system, just as if we had increased the stiffness of the spring. If we incorporate *velocity* feedback, we can increase the *damping* of the system.

Suppose that we attach a sensor whose output is proportional to the velocity of the car: $v_2 = K_v \dot{x}$ and feed it into the control current along with the position error

$$\begin{aligned}
i &= G(K_x(x_{des} - x) - K_v \dot{x}) \\
&= A_x x_{des} - A_x x - A_v \dot{x}
\end{aligned}$$

Taking Laplace transforms

$$I(s) = A_x X_{des}(s) - A_x X(x) - A_v s X(s)$$

and

$$ms^2 X(s) + (c + \frac{K_T A_v}{r}) s X(s) + (k + \frac{K_T A_x}{r}) X(s) = \frac{K_T A_x}{r} X_{des}(s) - W(s)$$

Defining k_1 as before and $c_1 = (c + \frac{K_T A_v}{r})$ we get

$$\begin{aligned}
\omega_n &= \sqrt{\frac{k_1}{m}} \\
\alpha &= \frac{c_1}{2m} \\
\zeta &= \frac{c_1}{2\sqrt{mk_1}}
\end{aligned}$$

so that we can now control both the stiffness and damping of the system by adjusting A_x and A_v.

The idea of *feeding back* measurements of the state of a system, comparing them with the desired state, and generating appropriate signals to drive the actuators can be generalized to any system where we want a mechanical output to correspond to an electrical input, as shown in Figure 2.8. Such systems are sometimes called *servomechanisms*. With a little imagination this can be seen as an instance of the generic electromechanical system shown in Figure 1.2.

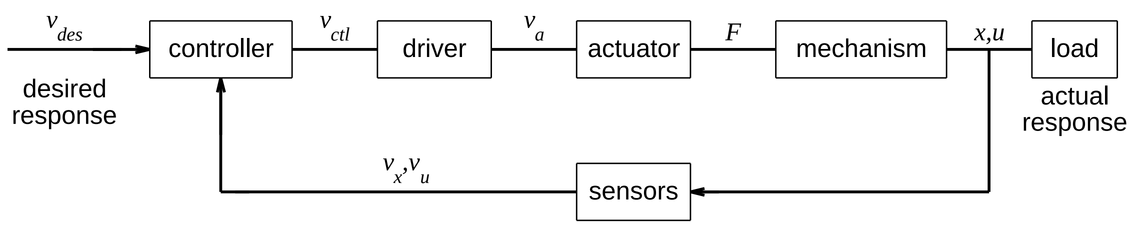

Figure 2.8: Generalized closed-loop control system