Review Problems for Exam 3

1. **Wiener Filtering:**

   Consider the simple problem of estimating a signal $s[n]$ observed in noise $w[n]$. That is, we observe $x[n] = s[n] + w[n]$ and want to design a filter $h[n]$ that minimizes the MSE $E[(s[n] - h[n] * x[n])^2]$. Recall that in this case the optimal Wiener filter is expressed in the frequency domain as

   $$H(\omega) = \frac{S_{ss}(\omega)}{S_{ss}(\omega) + S_{ww}(\omega)},$$

   where $S_{ss}(\omega)$ is the power spectral density of the signal $s[n]$ and $S_{ww}(\omega)$ is the power spectral density of the noise $w[n]$.

   Derive an expression for the mean-square error of the optimal Wiener filter at each frequency. That is, determine $E[|S(\omega) - H(\omega)X(\omega)|^2]$, where $S(\omega)$ and $X(\omega)$ are the DTFTs of $s[n]$ and $x[n]$, respectively. **HINT:** If $y[n]$ is a stationary random process, then the power spectral density $S_{yy}(\omega) = E[Y(\omega)Y^*(\omega)]$.

2. Adaptive Nonlinear System Identification

Suppose that we are trying to model a nonlinear DT system, depicted below, based on measurements of the input $x[n]$ and output $y[n]$.

Let us adopt the following nonlinear DT filter as our model

$$\widehat{y}[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n] x[n-1].$$

Notice the nonlinear cross-product term $b_2 x[n] x[n-1]$. To model the system we need to determine the values for the filter parameters b_0, b_1, b_2 that minimize the error between the true output $y[n]$ and the output of the filter $\widehat{y}[n]$.

a. Devise an LMS type algorithm for adaptively estimating the parameters. Give explicit expressions for the recursive parameter update formulae.

b. Suppose that the input $x[n]$ is zero-mean, Gaussian white noise with variance σ^2. The convergence behavior of this LMS algorithm depends on the covariance matrix $\mathbf{R}_{xx} = E[\mathbf{x}_n \mathbf{x}_n^T]$, where $\mathbf{x}_n = [x[n],\ x[n-1],\ x[n]x[n-1]]^T$. Find an expression for $\mathbf{R}_{xx}$ in terms of σ^2.

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2. (continued)

c. Based on the covariance matrix you determined above, suggest a range for the step size μ that should ensure stability.

d. Now consider a slightly more complex filter

$$\hat{y}[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n]x[n-1] + b_3 x^2[n] + b_4 x^2[n-1].$$

Also, assume that the input is a binary random signal, with each $x[n]$ independent and taking values ± 1 with equal probability. Determine $\mathbf{R}_{xx} = E[\mathbf{x}_n \mathbf{x}_n^T]$ in this case, where $\mathbf{x}_n = [x[n],\ x[n-1],\ x[n]x[n-1],\ x^2[n],\ x^2[n-1]]^T$. What range of step sizes would you recommend?

3. Adaptive De-Jammer

We are receiving a (sampled) sinusoidal radio signal at frequency ω_0. Unfortunately, an evil madman is going to try to disrupt our listening enjoyment by tranmitting a strong sinusoidal interference signal at another frequency. The madman is really twisted, and we never know what frequency he will be using to jam us, but *we do know that the frequency he will use is different than* ω_0.

Let $x[n]$ denote the signal we receive (the radio signal we are interested in at frequency ω_0 plus the madman's jamming signal). Here is our plan of action. We will:

- Design an FIR filter $h[n]$ to "null-out" the desired radio signal at ω_0, so that the filter output $y[n] = x[n] * h[n]$ contains only the jamming signal.
- Use LMS to adaptively tune another FIR filter $g[n]$ to "null-out" the jamming signal in $y[n]$.
- Apply the resulting filter $g[n]$ to the received signal $x[n]$.

The plan is analyzed in the following exercises: **(a)**, **(b)**, and **(c)**.

(a) First, design an FIR notch filter $h[n]$ to "null-out" the radio signal at ω_0, by specifying the location of the zeros. Express the filter's impulse response in terms of ω_0.

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3. (continued)

(b) Pass the received signal $x[n]$ through the filter $h[n]$. What remains is the madman's jamming signal at an unknown frequency. Let $y[n]$ denote the output of the filter $h[n]$ and devise an LMS filtering algorithm to adaptively adjust another FIR filter $g[n]$ to eliminate the jamming signal. **HINT:** The goal here is to drive $y[n]$ to zero. That is, we want to adapt $g[n]$ so that $g[n] * y[n] \approx 0$. Derive the LMS update equation for this problem.

(c) Now apply the adaptively tuned filter $g[n]$ to the original received signal $x[n]$. Explain the overall effect of this system. Contrast the performance of this adaptive filter with that of a fixed bandpass filter at ω_0. **HINT:** Consider the case if ω_1 is very close to ω_0.

4. Wiener Filtering and Denoising

Recall that Wiener filtering can be employed with any orthogonal signal transformation. The DFT is the usual basis for Wiener filtering, but we could also use the discrete wavelet transform (DWT) or another orthogonal signal transform.

Let $\theta_0, \ldots, \theta_{N-1}$ denote the coefficients resulting from an orthogonal transform of a finite duration input signal $s[n]$, $n = 0, \ldots, N-1$. Suppose that we observe a noisy version of the signal

$$x[n] = s[n] + w[n],$$

where $w[n]$, $n = 0, \ldots, N-1$, is a Gaussian white noise process with *known* variance σ^2. The orthogonal transform coefficients of $x[n]$ are

$$\omega_k = \theta_k + w_k, \; k = 0, \ldots, N-1,$$

where w_k are the transform coefficients of the noise process $w[n]$. Assume that the signal $s[n]$ is a realization of a zero-mean random process. In class we derived the optimal Wiener filter for each coefficient:

$$\widehat{\theta}_k = \frac{E[\theta_k^2]}{E[\theta_k^2] + \sigma^2} \, \omega_k.$$

a. Assume that the autocorrelation function for the signal $s[n]$ is known and denote it by $R_{ss}[m]$. Let $\phi_k[n]$ denote the basis function associated with the coefficient θ_k; that is, $\theta_k = \sum_{n=0}^{N-1} \phi_k[n] s[n]$. Derive an expression for $E[\theta_k^2]$ in terms of $\phi_k[n]$ and $R_{ss}[m]$.

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4. (continued)

b. Let $R_{ss}[m] = A^2\delta[m]$. Derive an exact expression for $E[\theta_k^2]$ in terms of A.

c. Suppose that $R_{ss}[m]$ is not known. Give an expression for an estimate $\widehat{R}_{ss}[m]$ based on the noisy signal $x[n]$. **HINT:** Remember that $w[n]$ is Gaussian white noise.

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4. (continued)

d. Instead of estimating the autocorrelation function, we could just try to estimate the expected values $E[\theta_k^2]$. A reasonable estimate is $\omega_k^2 - \sigma^2$. Show that the expectation of $\omega_k^2 - \sigma^2$ is equal to $E[\theta_k^2]$.

e. We can construct an "approximate" Wiener filter by replacing $E[\theta_k^2]$ by $\omega_k^2 - \sigma^2$. One problem with this is that $E[\theta_k^2] \geq 0$ whereas $\omega_k^2 - \sigma^2$ may be negative. This negativity problem can be "fixed" by using

$$(\omega_k^2 - \sigma^2)_+ = \begin{cases} \omega_k^2 - \sigma^2 & , \quad \text{if } \omega_k^2 - \sigma^2 > 0 \\ 0 & , \quad \text{if } \omega_k^2 - \sigma^2 \leq 0 \end{cases}$$

instead. Let's use $(\omega_k^2 - \sigma^2)_+$ in place of $E[\theta_k^2]$ in the Wiener filter equation:

$$\widehat{\theta}_k = \frac{(\omega_k^2 - \sigma^2)_+}{(\omega_k^2 - \sigma^2)_+ + \sigma^2} \, \omega_k .$$

Sketch the input-output characteristic of this filter; that is, plot the $\widehat{\theta_k}$ versus ω_k. Describe the processing action of this filter. Is it linear?