Signals & Systems: Discrete-Time

ELEC 301

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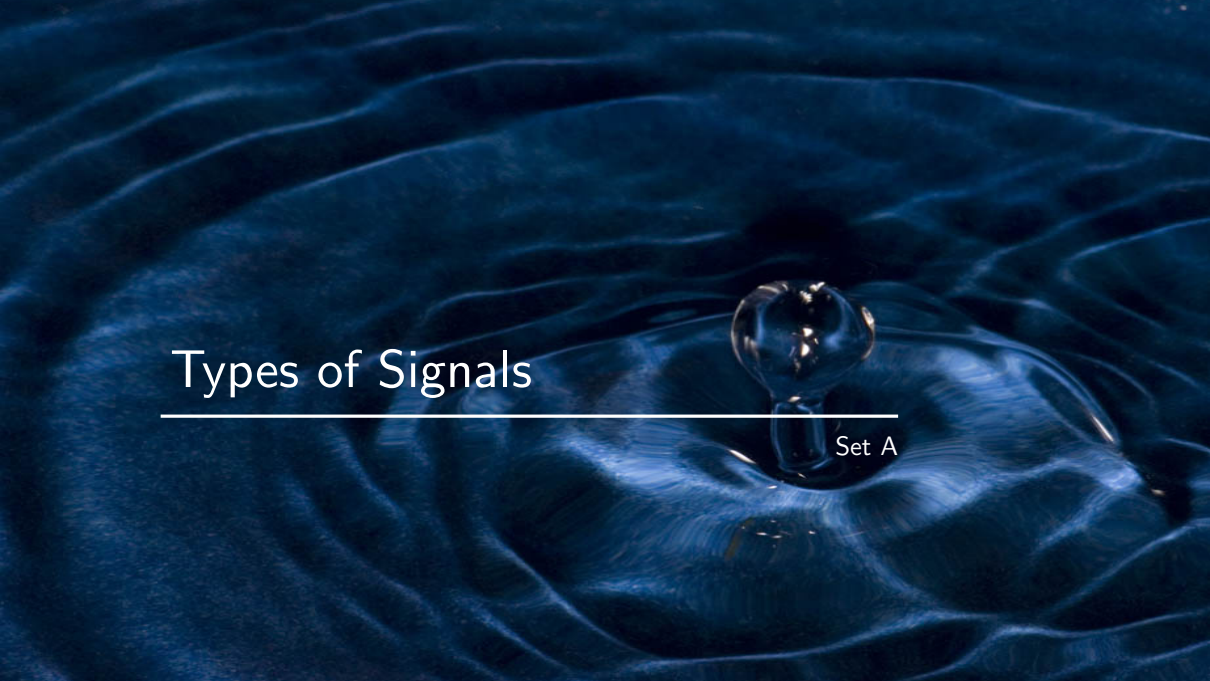
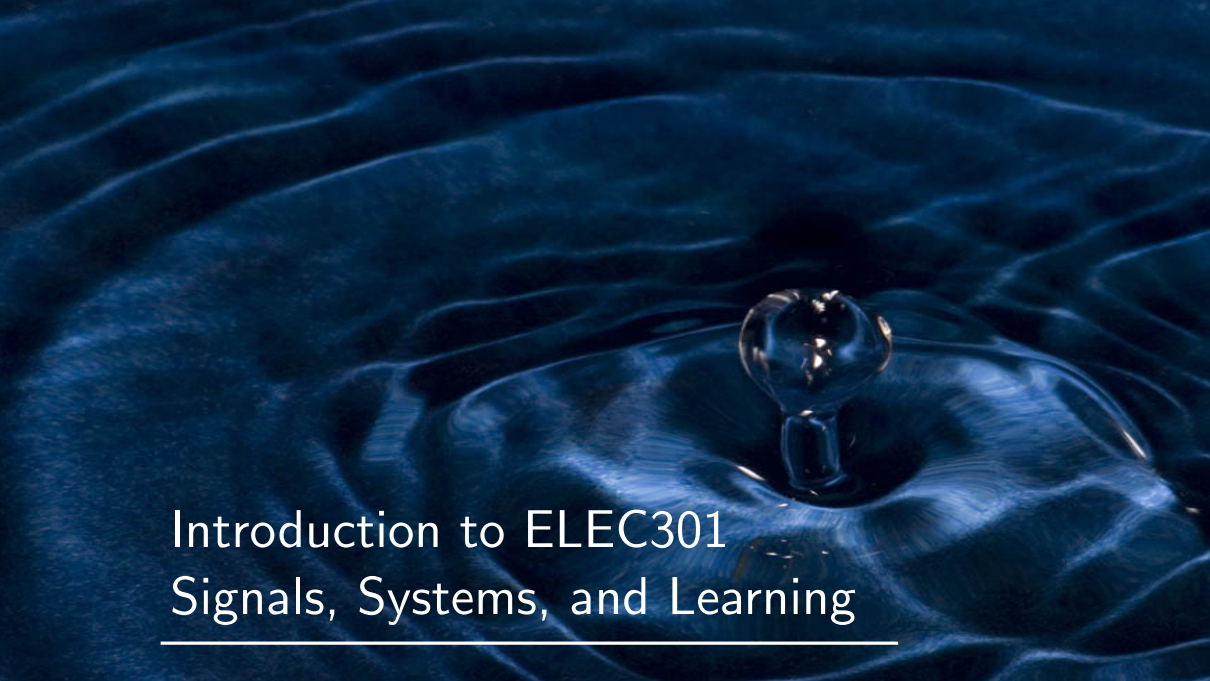


Types of Signals

Set A



Introduction to ELEC301
Signals, Systems, and Learning

Welcome to Elec301 – Signals, Systems, and Learning

- This is an introductory course on **signal processing**, which studies **signals**, **systems**, and **learning algorithms**

DEFINITION

Signal (n): A detectable physical quantity . . . by which messages or information can be transmitted (Merriam-Webster)

- Signals carry **information**
- Examples:
  - Speech signals transmit language via acoustic waves
  - Radar signals transmit the position and velocity of targets via electromagnetic waves
  - Electrophysiology signals transmit information about processes inside the body
  - Financial signals transmit information about events in the economy

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- **Systems** manipulate the information carried by signals

DEFINITION

Signal processing involves the theory and application of

- filtering, coding, transmitting, estimating, detecting, analyzing, recognizing, synthesizing, recording, and reproducing signals by digital or analog devices or techniques
- where signal includes audio, video, speech, image, communication, geophysical, sonar, radar, medical, musical, and other signals

(IEEE Signal Processing Society Constitutional Amendment, 1994)

Welcome to Elec301 – Signals, Systems, and Learning

- **Systems** manipulate the information carried by signals
- Broadly speaking, there are two ways to **design systems** (and we will study both)
- Classical approach, using **physics**
  - Pro: Small number of parameters (if any)
  - Con: Performance limited by the analytical powers of physicists
- Modern approach, by **learning** from (training) **data**
  - Con: Large number of parameters
  - Pro: Performance limited only by the amount of training data

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- **Learning algorithms** optimize how a system manipulates signals

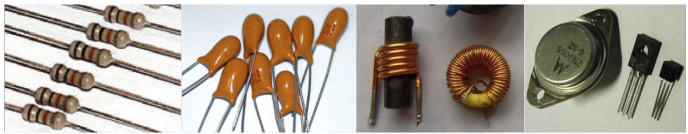
DEFINITION

"A computer program is said to **learn from experience** E with respect to some class of tasks T and performance measure P if its performance at tasks in T, as measured by P, improves with experience E" (Tom Mitchell, CMU)

- The field of **machine learning** is often associated with "**artificial intelligence**" (AI) and "**data science**"

Signal Processing

- Signal processing has traditionally been a part of electrical and computer engineering
- But now expands into applied mathematics, statistics, computer science, and a host of application disciplines
- Initially **analog** signals and systems implemented using resistors, capacitors, inductors, and transistors (Elec241/242)



- Since the 1940s increasingly **digital** signals and systems implemented using computers and computer code (Matlab, Python, C, ...)
  - Advantages of digital include stability and programmability
  - As computers have shrunk, digital signal processing has become ubiquitous
  - Digital processing can be tuned using training data (machine learning)

Signal Processing Applications

Rice ELEC301

- Goals: Develop intuition into and learn how to reason analytically about signal processing and machine learning problems
- Video lectures (first half of course), primary sources, supplemental materials, practice exercises, homework, coding exercises, group competition, final exam
- Coding in Matlab or Python
- **Important:** This is a mathematical treatment of signals, systems, and learning (no pain, no gain!)

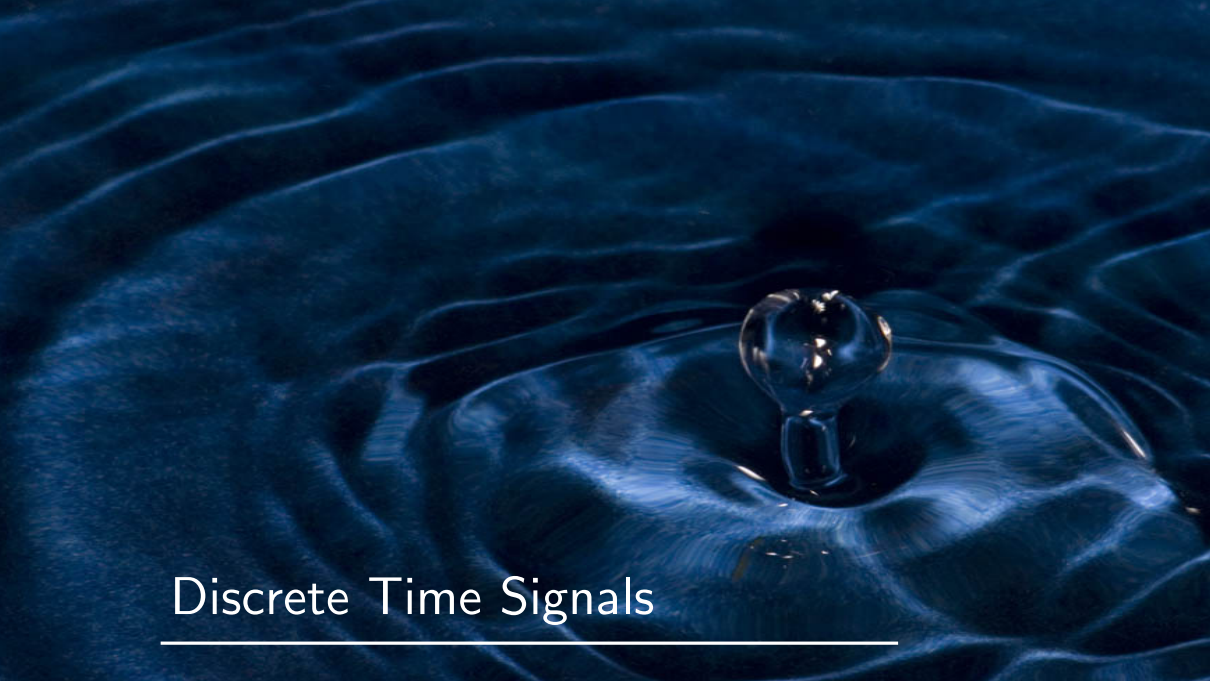
Before We Start

- Please make sure you have a solid understanding of
  - Complex numbers and arithmetic
  - Linear algebra (vectors, matrices, dot products, eigenvectors and eigenvalues, singular value decomposition (SVD), bases ...)
  - Sequences and series (finite and infinite)
  - Calculus of a single variable (derivatives and integrals)
  - Matlab or Python
- To test your readiness or refresh your knowledge, visit the "Pre-class Mathematics Refresher"

Administrata

- Course website: `http://dsp.rice.edu/courses/elec301`
- Course syllabus: `http://dsp.rice.edu/courses/elec301/syllabus`
- Piazza: `https://piazza.com/rice/fall2016/elec301/home`

Welcome to Elec301 – Signals, Systems, and Learning



Discrete Time Signals

Signals

DEFINITION

Signal (n): A detectable physical quantity . . . by which messages or information can be transmitted (Merriam-Webster)

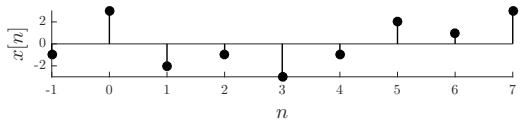
- Signals carry **information**
- Examples:
  - Speech signals transmit language via acoustic waves
  - Radar signals transmit the position and velocity of targets via electromagnetic waves
  - Electrophysiology signals transmit information about processes inside the body
  - Financial signals transmit information about events in the economy
- **Signal processing systems** manipulate the information carried by signals
- This is a course about signals and systems

Signals are Functions

DEFINITION

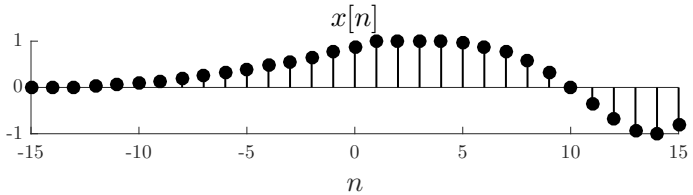
A **signal** is a function that maps an independent variable to a dependent variable.

- Signal $x[n]$: each value of n produces the value $x[n]$
- In this course, we will focus on **discrete-time** signals:
  - Independent variable is an **integer**: $n \in \mathbb{Z}$ (will refer to as time)
  - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or $\mathbb{C}$

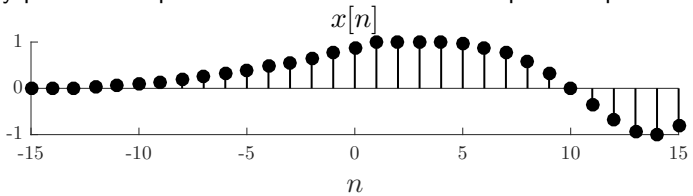


Plotting Real Signals

- When $x[n] \in \mathbb{R}$ (ex: temperature in a room at noon on Monday), we use one signal plot

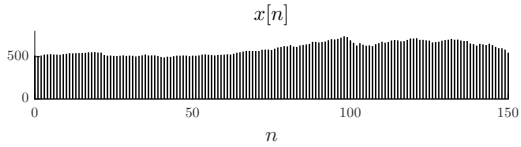


- We will typically place the dependent variable label to the top of the plot.

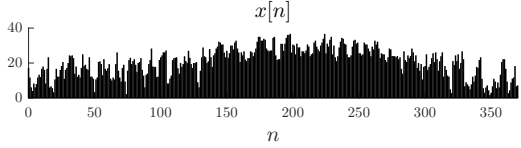


A Menagerie of Signals

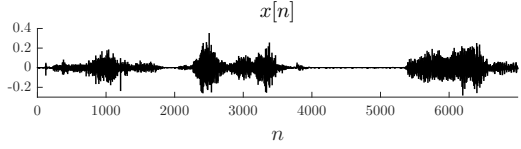
- Google Share daily share price for 5 months



- Temperature at Houston Intercontinental Airport in 2013 (Celcius)

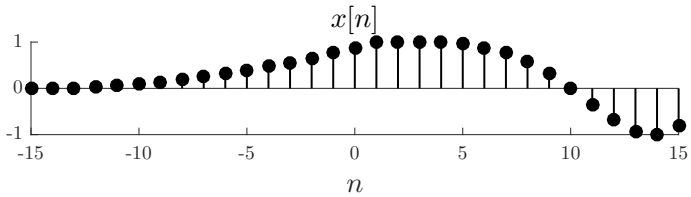


- Excerpt from Shakespeare's *Hamlet*

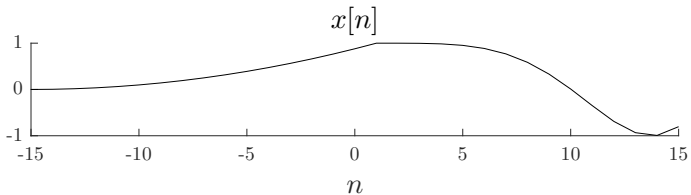


Plotting Signals Correctly

- In a discrete-time signal $x[n]$, the independent variable n is discrete (integer)
- To plot a discrete-time signal in a program like Matlab, you should use the `stem` or similar command and not the `plot` command
- Correct:



- Incorrect:

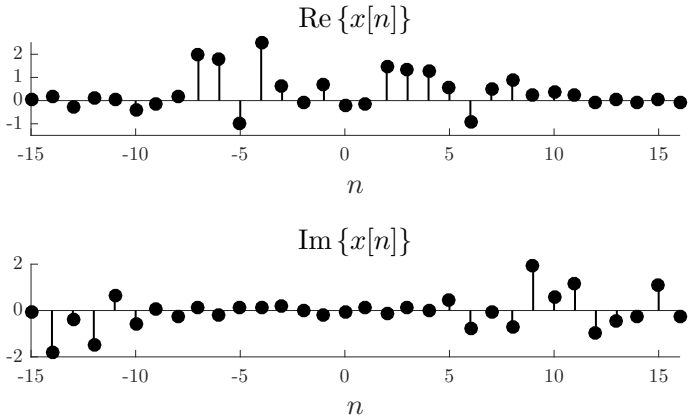


Plotting Complex Signals

- Recall that a complex number $a \in \mathbb{C}$ can be equivalently represented two ways:
  - Polar form: $a = |a|\, e^{j\angle a}$
  - Rectangular form: $a = \mathrm{Re}\{a\} + j\,\mathrm{Im}\{a\}$
- Here $j = \sqrt{-1}$ (engineering notation; mathematicians use $i = \sqrt{-1}$)
- When $x[n] \in \mathbb{C}$ (ex: magnitude and phase of an electromagnetic wave), we use two signal plots
  - Rectangular form: $x[n] = \mathrm{Re}\{x[n]\} + j\,\mathrm{Im}\{x[n]\}$
  - Polar form: $x[n] = |x[n]|\, e^{j\angle x[n]}$

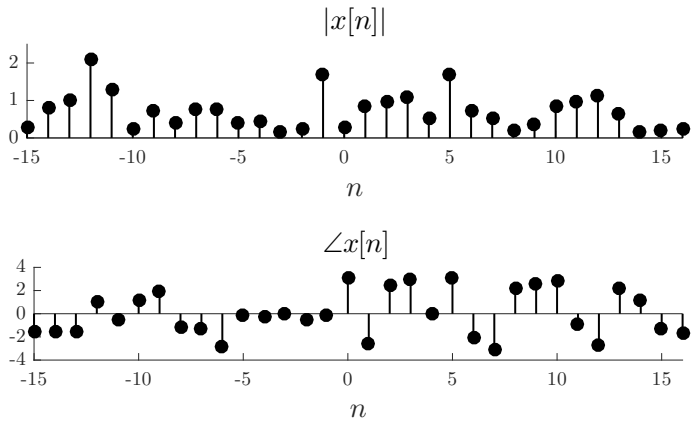
Plotting Complex Signals (Rectangular Form)

- Rectangular form: $x[n] = \mathrm{Re}\{x[n]\} + j\,\mathrm{Im}\{x[n]\} \in \mathbb{C}$



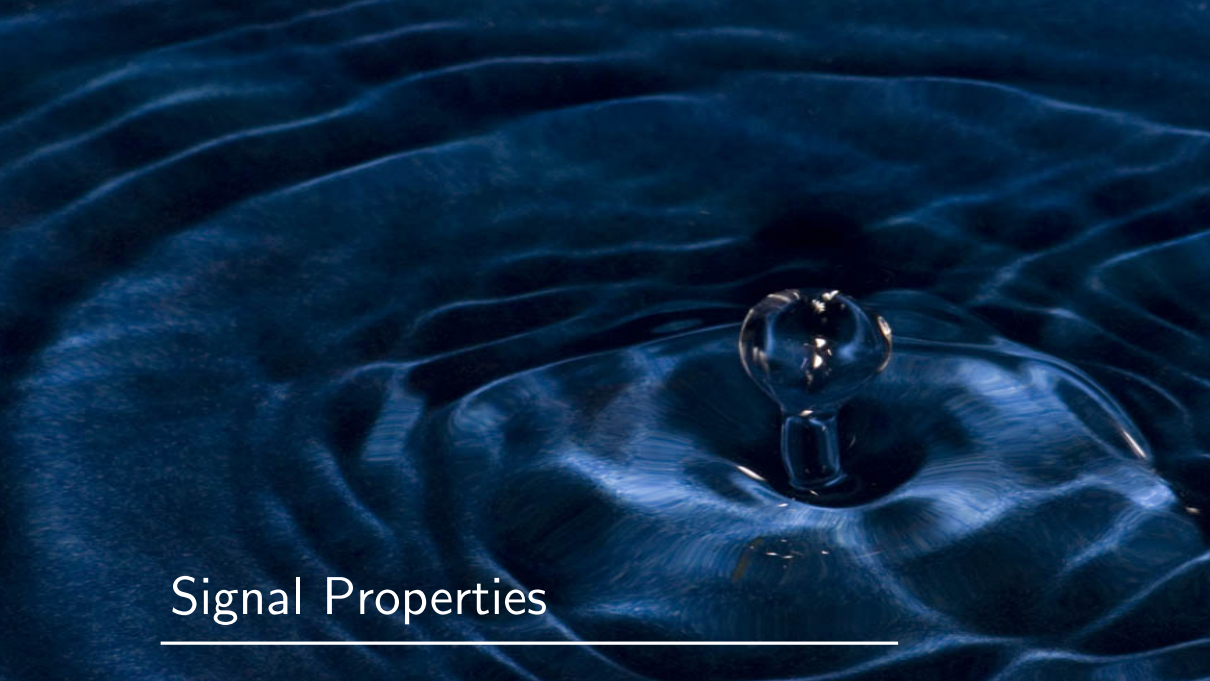
Plotting Complex Signals (Polar Form)

- Polar form: $x[n] = |x[n]|\, e^{j\angle(x[n])} \in \mathbb{C}$



Summary

- Discrete-time signals
  - Independent variable is an integer: $n \in \mathbb{Z}$ (will refer to as time)
  - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or $\mathbb{C}$
- Plot signals correctly!



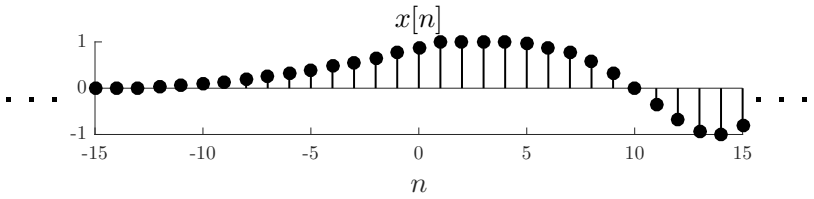
Signal Properties

Signal Properties

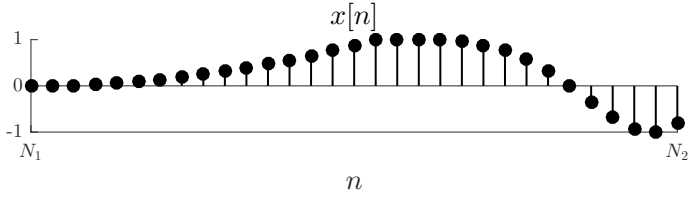
- Infinite/finite-length signals
- Periodic signals
- Causal signals
- Even/odd signals
- Digital signals

Finite/Infinite-Length Signals

- An **infinite-length** discrete-time signal $x[n]$ is defined for all $n \in \mathbb{Z}$, i.e., $-\infty < n < \infty$



- A **finite-length** discrete-time signal $x[n]$ is defined only for a finite range of $N_1 \leq n \leq N_2$



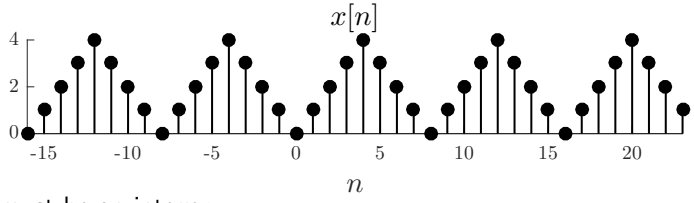
- Important: a finite-length signal is undefined for $n < N_1$ and $n > N_2$

Periodic Signals

DEFINITION

A discrete-time signal is **periodic** if it repeats with period $N \in \mathbb{Z}$:

$$x[n + mN] = x[n] \quad \forall\, m \in \mathbb{Z}$$



Notes:

- The period N must be an integer
- A periodic signal is infinite in length

DEFINITION

A discrete-time signal is **aperiodic** if it is not periodic

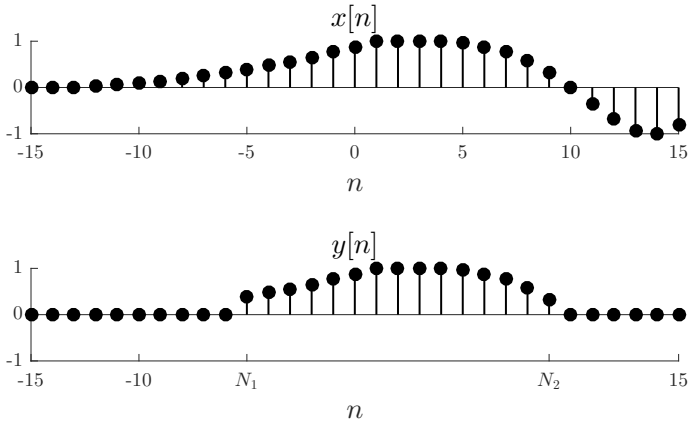
Converting between Finite and Infinite-Length Signals

- Convert an infinite-length signal into a finite-length signal by **windowing**
- Convert a finite-length signal into an infinite-length signal by either
  - (infinite) **zero padding**, or
  - **periodization**

Windowing

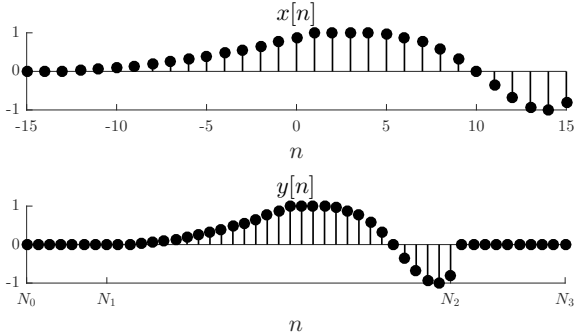
- Converts a longer signal into a shorter one

$$y[n] = \begin{cases} x[n] & N_1 \leq n \leq N_2 \\ 0 & \text{otherwise} \end{cases}$$



Zero Padding

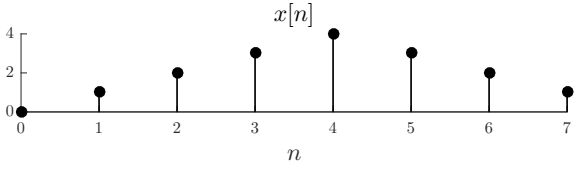
- Converts a shorter signal into a longer one
- Say $x[n]$ is defined for $N_1 \leq n \leq N_2$
- Given $N_0 \leq N_1 \leq N_2 \leq N_3$ $\quad y[n] = \begin{cases} 0 & N_0 \leq n < N_1 \\ x[n] & N_1 \leq n \leq N_2 \\ 0 & N_2 < n \leq N_3 \end{cases}$



Periodization

- Converts a finite-length signal into an infinite-length, periodic signal
- Given finite-length $x[n]$, replicate $x[n]$ periodically with period N

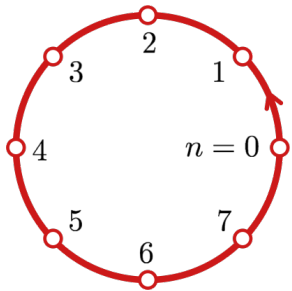
$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} x[n-mN], \quad n \in \mathbb{Z} \\ &= \cdots + x[n+2N] + x[n+N] + x[n] + x[n-N] + x[n-2N] + \cdots \end{aligned}$$



Useful Aside – Modular Arithmetic

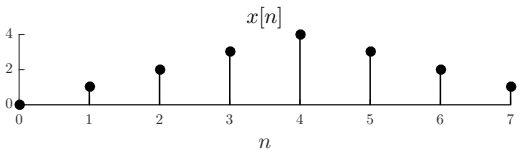
- Modular arithmetic with **modulus** N (mod-N) takes place on a **wheel** with N time locations
  - Ex: $(12)_8$ ("twelve mod eight")

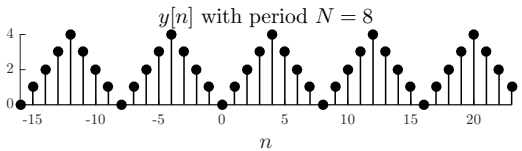
- Modulo arithmetic is inherently **periodic**
  - Ex: $\ldots\ (-12)_8 = (-4)_8 = (4)_8 = (12)_8 = (20)_8\ \ldots$



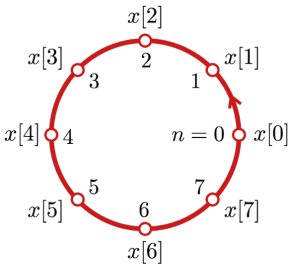
Periodization via Modular Arithmetic

- Consider a length-N signal $x[n]$ defined for $0 \leq n \leq N-1$
- A convenient way to express periodization with period N is $\quad y[n] = x[(n)_N], \quad n \in \mathbb{Z}$

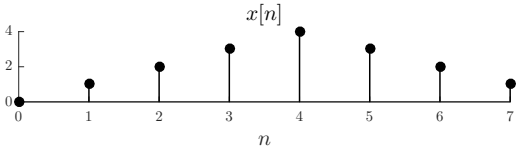


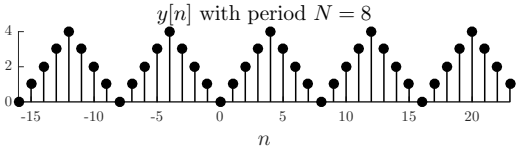


- Important interpretation
  - Infinite-length signals live on the (infinite) number **line**
  - Periodic signals live on a **circle** – a wheel with N time locations



Finite-Length and Periodic Signals are Equivalent



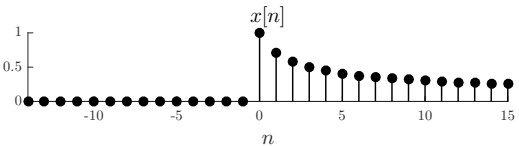


- All of the information in a periodic signal is contained in **one period** (of finite length)
- Any finite-length signal can be periodized
- Conclusion: We can and will think of finite-length signals and periodic signals interchangeably
- We can choose the most convenient viewpoint for solving any given problem
  - Application: Shifting finite length signals

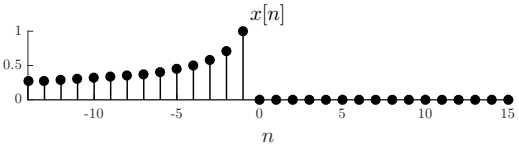
Causal Signals

DEFINITION

A signal $x[n]$ is **causal** if $x[n] = 0$ for all $n < 0$.



- A signal $x[n]$ is **anti-causal** if $x[n] = 0$ for all $n \geq 0$

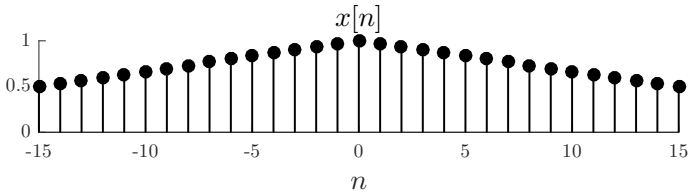


- A signal $x[n]$ is **acausal** if it is not causal

Even Signals

DEFINITION

A real signal $x[n]$ is **even** if $x[-n] = x[n]$

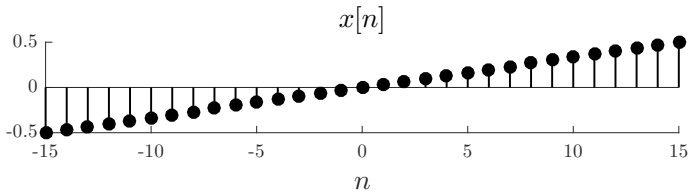


- Even signals are symmetrical around the point $n = 0$

Odd Signals

DEFINITION

A real signal $x[n]$ is **odd** if $x[-n] = -x[n]$



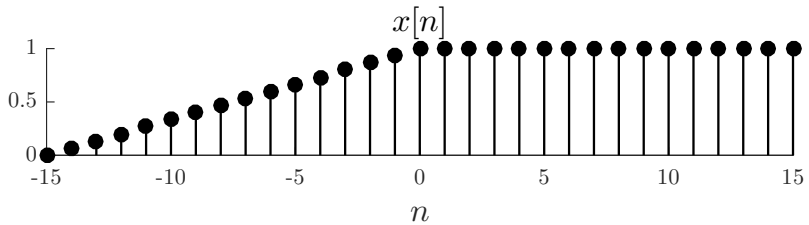
- Odd signals are anti-symmetrical around the point $n = 0$

Even+Odd Signal Decomposition

- **Useful fact:** Every signal $x[n]$ can be decomposed into the sum of its even part + its odd part
- Even part: $e[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$ (easy to verify that $e[n]$ is even)
- Odd part: $o[n] = \frac{1}{2}\left(x[n] - x[-n]\right)$ (easy to verify that $o[n]$ is odd)
- **Decomposition** $x[n] = e[n] + o[n]$
- Verify the decomposition:

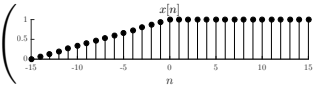
$$\begin{aligned} e[n] + o[n] &= \frac{1}{2}(x[n] + x[-n]) + \frac{1}{2}(x[n] - x[-n]) \\ &= \frac{1}{2}(x[n] + x[-n] + x[n] - x[-n]) \\ &= \frac{1}{2}(2\,x[n]) = x[n] \quad \checkmark \end{aligned}$$

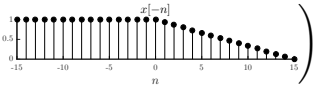
Even+Odd Signal Decomposition in Pictures

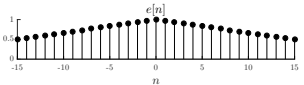


Even+Odd Signal Decomposition in Pictures

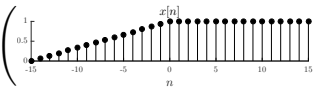
$$\frac{1}{2}\left(\right. \quad + \quad \left. \right) = \quad + \quad$$

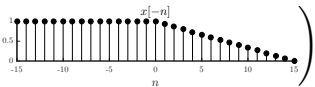


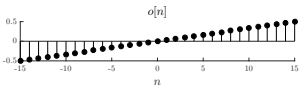


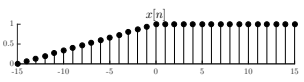


$$\frac{1}{2}\left(\right. \quad - \quad \left. \right) = \quad = \quad$$



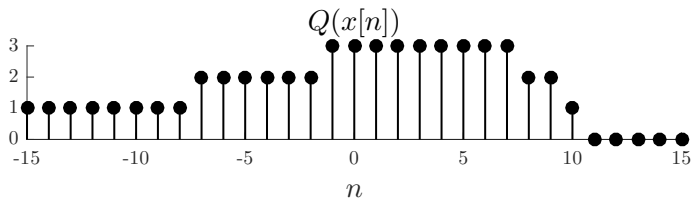






Digital Signals

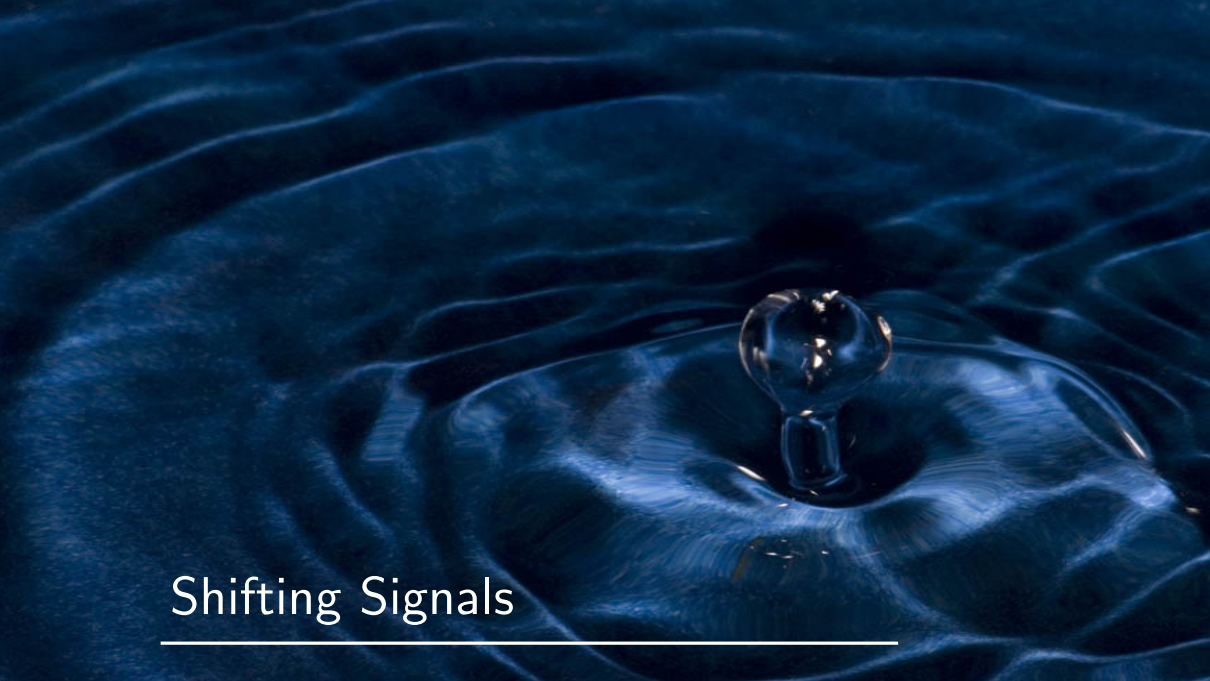
- **Digital signals** are a special sub-class of discrete-time signals
  - Independent variable is still an integer: $n \in \mathbb{Z}$
  - Dependent variable is from a **finite set of integers**: $x[n] \in \{0, 1, \ldots, D-1\}$
  - Typically, choose $D = 2^q$ and represent each possible level of $x[n]$ as a digital code with q bits
  - Ex: Digital signal with $q = 2$ bits $\Rightarrow D = 2^2 = 4$ levels



  - Ex: Compact discs use $q = 16$ bits $\Rightarrow D = 2^{16} = 65536$ levels

Summary

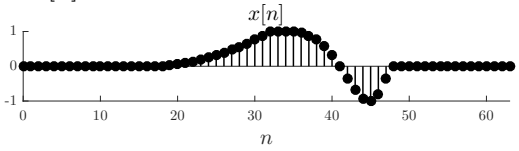
- Signals can be classified many different ways (real/complex, infinite/finite-length, periodic/aperiodic, causal/acausal, even/odd, ...)
- Finite-length signals are equivalent to periodic signals; modulo arithmetic useful



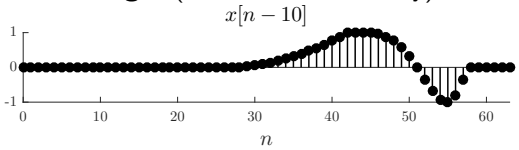
Shifting Signals

Shifting Infinite-Length Signals

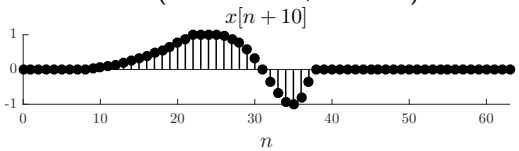
- Given an infinite-length signal $x[n]$, we can **shift** it back and forth in time via $x[n-m]$, $m \in \mathbb{Z}$



- When $m > 0$, $x[n-m]$ shifts to the **right** (forward in time, delay)



- When $m < 0$, $x[n-m]$ shifts to the **left** (back in time, advance)

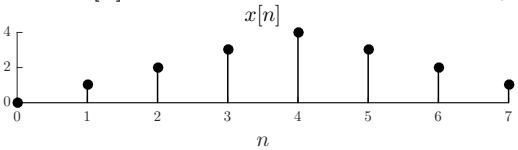


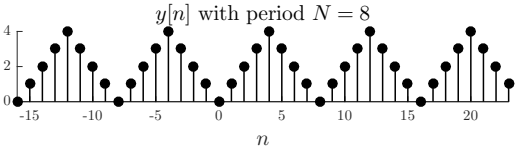
Periodic Signals and Modular Arithmetic

- A convenient way to express a signal $y[n]$ that is periodic with period N is

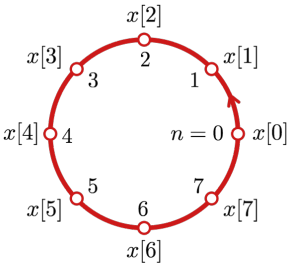
$$y[n] = x[(n)_N], \quad n \in \mathbb{Z}$$

where $x[n]$, defined for $0 \le n \le N-1$, comprises one period



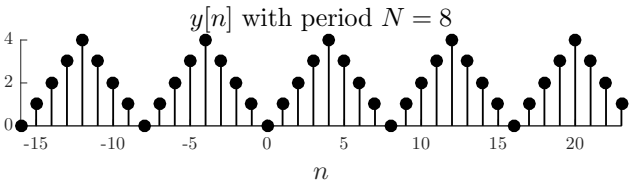


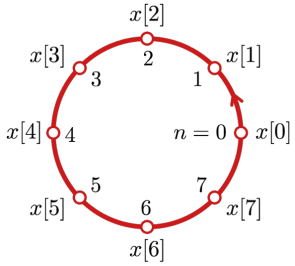
- Important interpretation
  - Infinite-length signals live on the (infinite) number line
  - Periodic signals live on a circle – a wheel with N time places



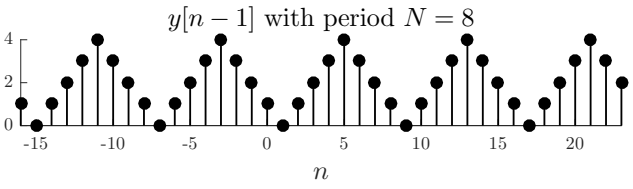
Shifting Periodic Signals

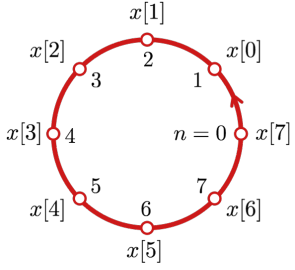
- Periodic signals can also be shifted; consider $y[n] = x[(n)_N]$





- Shift one sample into the future: $y[n-1] = x[(n-1)_N]$





Shifting Finite-Length Signals

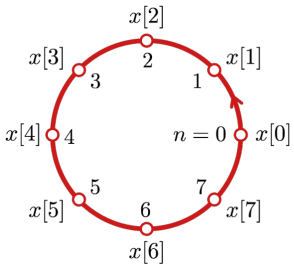
- Consider finite-length signals x and v defined for $0 \leq n \leq N-1$ and suppose "$v[n] = x[n-1]$"

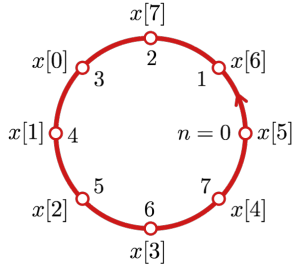
$$\begin{aligned} v[0] &= ?? \\ v[1] &= x[0] \\ v[2] &= x[1] \\ v[3] &= x[2] \\ &\vdots \\ v[N-1] &= x[N-2] \\ ?? &= x[N-1] \end{aligned}$$

- What to put in $v[0]$? What to do with $x[N-1]$? We don't want to invent/lose information
- Elegant solution: Assume x and v are both periodic with period N; then $v[n] = x[(n-1)_N]$
- This is called a **periodic** or **circular shift** (see `circshift` and `mod` in Matlab)

Circular Shift Example

- Elegant formula for circular shift of $x[n]$ by m time steps: $x[(n-m)_N]$
- Ex: x and v defined for $0 \leq n \leq 7$, that is, $N = 8$. Find $v[n] = x[(n-3)_8]$



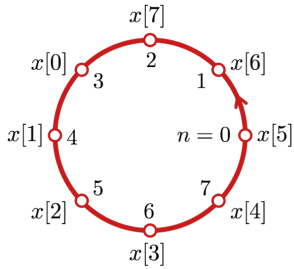


Circular Shift Example

- Elegant formula for circular shift of $x[n]$ by m time steps: $x[(n-m)_N]$

- Ex: x and v defined for $0 \le n \le 7$, that is, $N = 8$. Find $v[n] = x[(n-m)_N]$

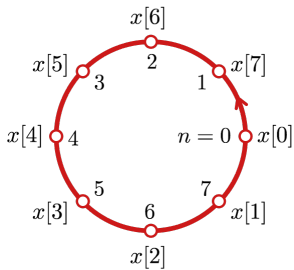
$$
\begin{aligned}
v[0] &= x[5] \\
v[1] &= x[6] \\
v[2] &= x[7] \\
v[3] &= x[0] \\
v[4] &= x[1] \\
v[5] &= x[2] \\
v[6] &= x[3] \\
v[7] &= x[4]
\end{aligned}
$$



Circular Time Reversal

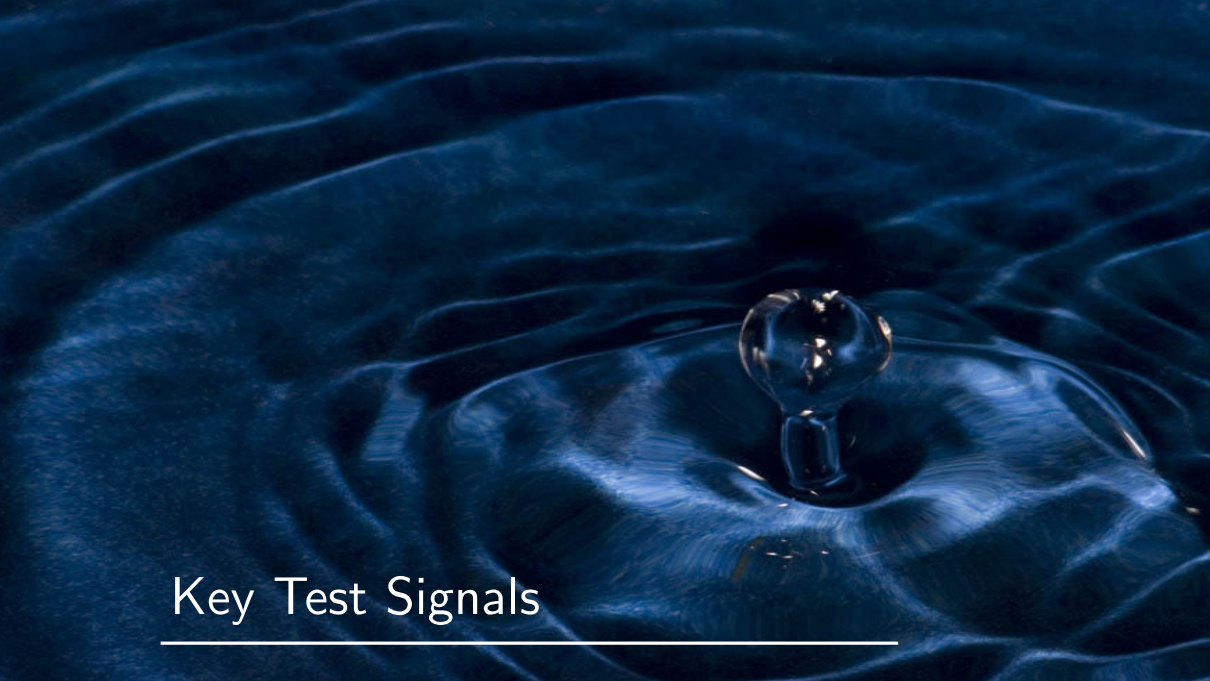
- For infinite length signals, the transformation of reversing the time axis $x[-n]$ is obvious
- Not so obvious for periodic/finite-length signals
- Elegant formula for reversing the time axis of a periodic/finite-length signal: $x[(-n)_N]$
- Ex: x and v defined for $0 \leq n \leq 7$, that is, $N = 8$. Find $v[n] = x[(-n)_N]$

$$
\begin{aligned}
v[0] &= x[0] \\
v[1] &= x[7] \\
v[2] &= x[6] \\
v[3] &= x[5] \\
v[4] &= x[4] \\
v[5] &= x[3] \\
v[6] &= x[2] \\
v[7] &= x[1]
\end{aligned}
$$



Summary

- Shifting a signal moves it forward or backward in time
- Modulo arithmetic provides and easy way to shift periodic signals



Key Test Signals

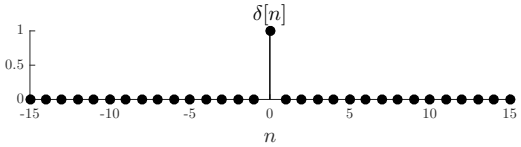
A Toolbox of Test Signals

- Delta function
- Unit step
- Unit pulse
- Real exponential
- Still to come: sinusoids, complex exponentials
- **Note:** We will introduce the test signals as infinite-length signals, but each has a finite-length equivalent

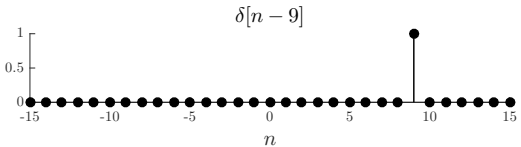
Delta Function

DEFINITION

The **delta function** (aka unit impulse) $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$



- The shifted delta function $\delta[n-m]$ peaks up at $n = m$; here $m = 9$

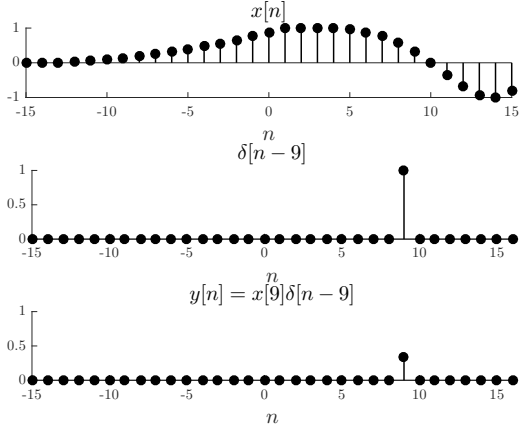


Delta Functions Sample

- Multiplying a signal by a shifted delta function picks out one sample of the signal and sets all other samples to zero

$$y[n] \;=\; x[n]\,\delta[n-m] \;=\; x[m]\,\delta[n-m]$$

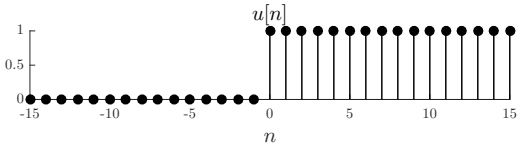
- Important: m is a fixed constant, and so $x[m]$ is a constant (and not a signal)



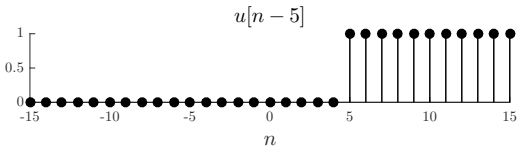
Unit Step

DEFINITION

The **unit step** $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$



- The shifted unit step $u[n-m]$ jumps from 0 to 1 at $n = m$; here $m = 5$

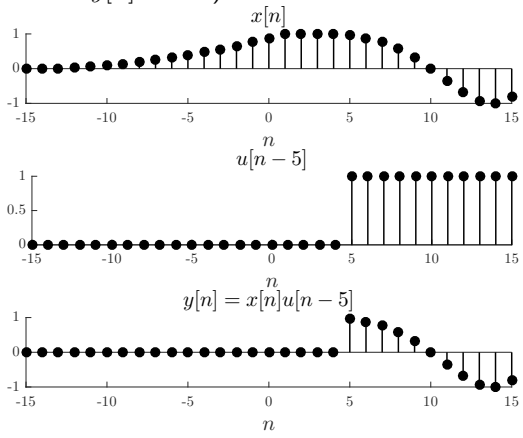


Unit Step Selects Part of a Signal

- Multiplying a signal by a shifted unit step function zeros out its entries for $n < m$

$$y[n] \;=\; x[n]\,u[n-m]$$

(Note: For $m = 0$, this makes $y[n]$ causal)

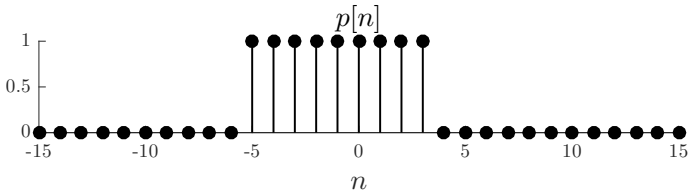


Unit Pulse

DEFINITION

The **unit pulse** (aka boxcar) $$p[n] = \begin{cases} 0 & n < N_1 \\ 1 & N_1 \le n \le N_2 \\ 0 & n > N_2 \end{cases}$$

- Ex: $p[n]$ for $N_1 = -5$ and $N_2 = 3$



- One of many different formulas for the unit pulse

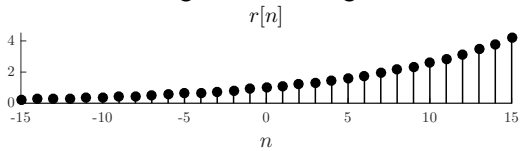
$$p[n] = u[n - N_1] - u[n - (N_2 + 1)]$$

Real Exponential

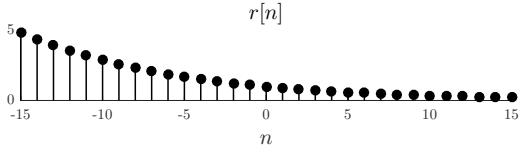
DEFINITION

The **real exponential** $r[n] = a^n,\ a \in \mathbb{R},\ a \geq 0$

- For $a > 1$, $r[n]$ shrinks to the left and grows to the right; here $a = 1.1$



- For $0 < a < 1$, $r[n]$ grows to the left and shrinks to the right; here $a = 0.9$



Summary

- We will use our test signals often, especially the delta function and unit step

Sinusoids

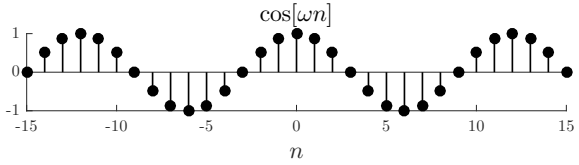
Sinusoids

- Sinusoids appear in myriad disciplines, in particular signal processing
- They are the basis (literally) of Fourier analysis (DFT, DTFT)
- We will introduce
  - Real-valued sinusoids
  - (Complex) sinusoid
  - Complex exponential

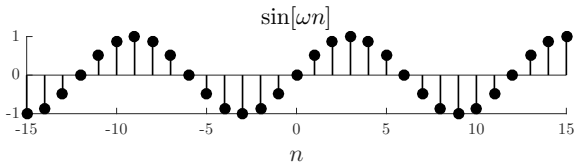
Sinusoids

- There are two natural real-valued sinusoids: $\cos(\omega n + \phi)$ and $\sin(\omega n + \phi)$
- **Frequency:** ω (units: radians/sample)
- **Phase:** ϕ (units: radians)

- $\cos(\omega n)$ (even)

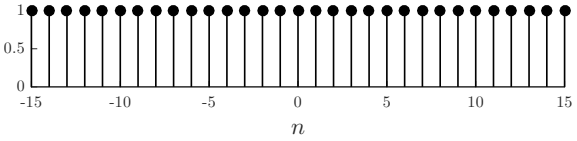


- $\sin(\omega n)$ (odd)

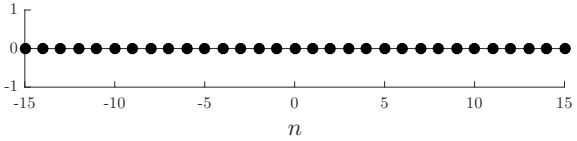


Sinusoid Examples

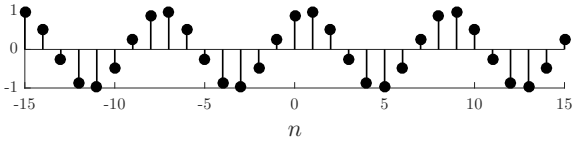
- $\cos(0n)$



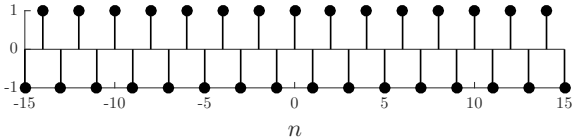
- $\sin(0n)$



- $\sin(\frac{\pi}{4}n + \frac{2\pi}{6})$



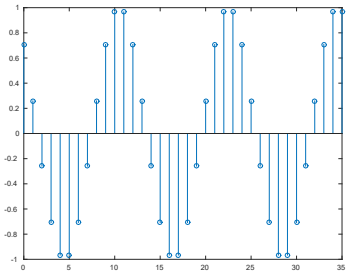
- $\cos(\pi n)$



Get Comfortable with Sinusoids!

- It's easy to play around in Matlab to get comfortable with the properties of sinusoids

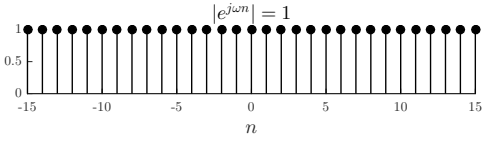
```
N=36;
n=0:N-1;
omega=pi/6;
phi=pi/4;
x=cos(omega*n+phi);
stem(n,x)
```

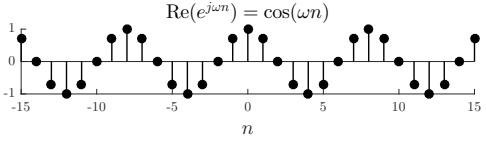


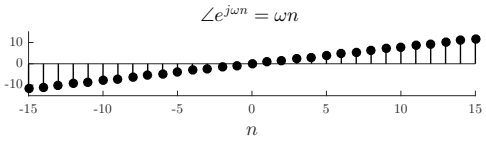
Complex Sinusoid

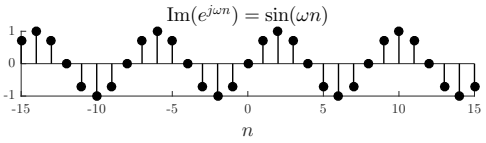
- The complex-valued sinusoid combines both the $\cos$ and $\sin$ terms (via Euler's identity)

$$e^{j(\omega n+\phi)} = \cos(\omega n + \phi) + j\sin(\omega n + \phi)$$



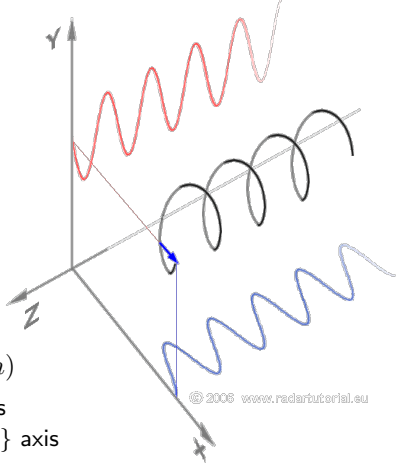






A Complex Sinusoid is a Helix

$$e^{j(\omega n+\phi)} = \cos(\omega n + \phi) + j\sin(\omega n + \phi)$$



- A complex sinusoid is a **helix** in 3D space $(\mathrm{Re}\{\}, \mathrm{Im}\{\}, n)$
  - Real part ($\cos$ term) is the projection onto the $\mathrm{Re}\{\}$ axis
  - Imaginary part ($\sin$ term) is the projection onto the $\mathrm{Im}\{\}$ axis
- Frequency ω determines rotation speed and direction of helix
  - $\omega > 0 \Rightarrow$ anticlockwise rotation
  - $\omega < 0 \Rightarrow$ clockwise rotation

Complex Sinusoid is a Helix (Animation)

- **Complex sinusoid animation**

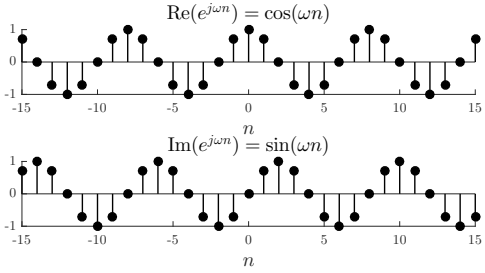
Negative Frequency

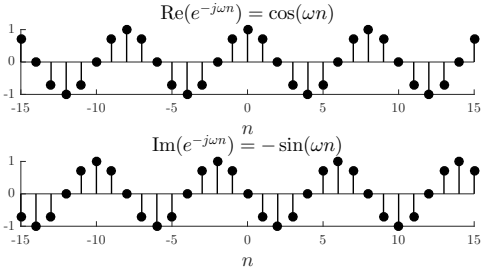
- Negative frequency is nothing to be afraid of! Consider a sinusoid with a negative frequency $-\omega$

$$e^{j(-\omega)n} \;=\; e^{-j\omega n} \;=\; \cos(-\omega n) + j\sin(-\omega n) \;=\; \cos(\omega n) - j\sin(\omega n)$$

- Also note: $e^{j(-\omega)n} \;=\; e^{-j\omega n} \;=\; \left(e^{j\omega n}\right)^*$

- Bottom line: negating the frequency is equivalent to complex conjugating a complex sinusoid, which flips the sign of the imaginary, $\sin$ term



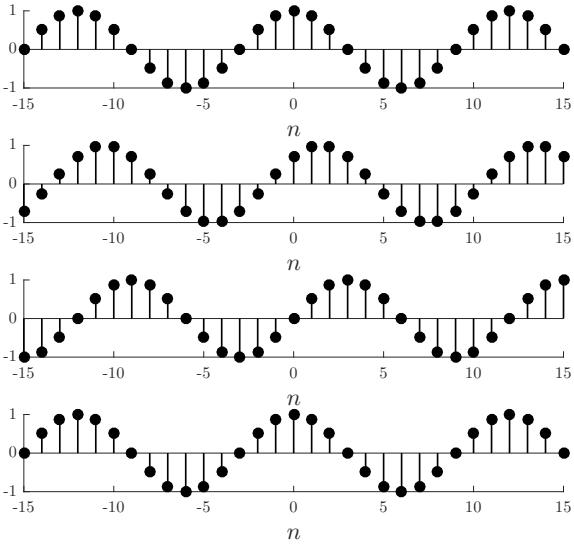


Phase of a Sinusoid

$e^{j(\omega n+\phi)}$

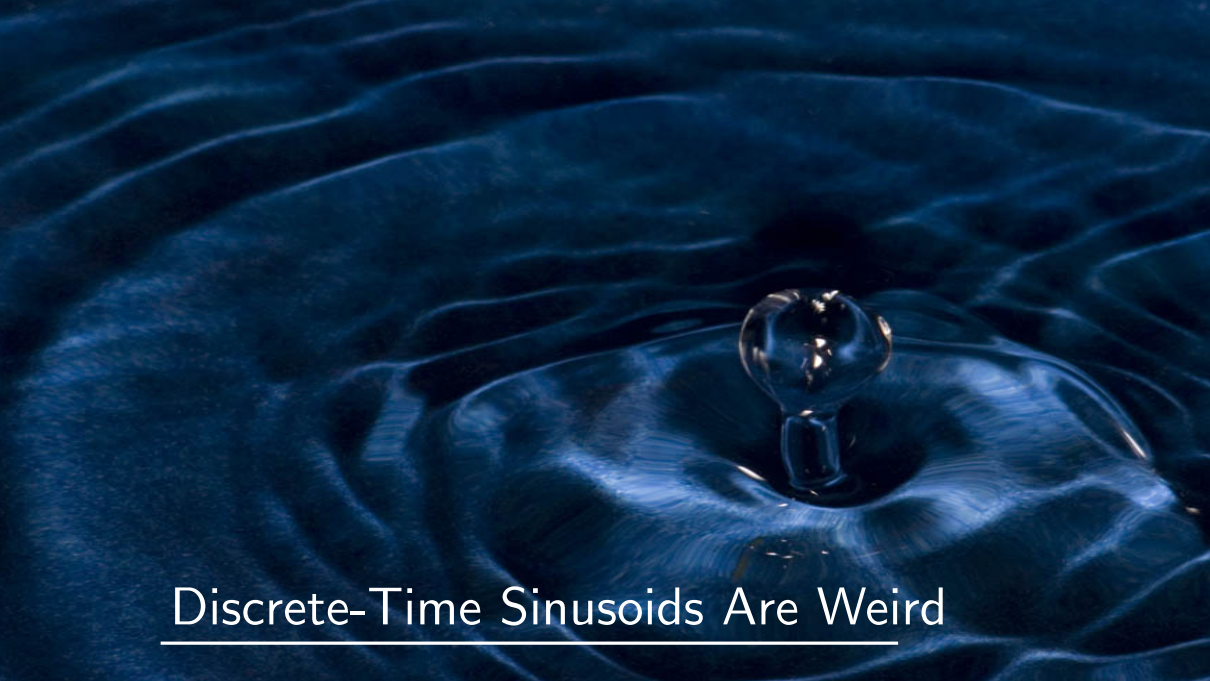
- ϕ is a (frequency independent) shift that is referenced to one period of oscillation

- $\cos\left(\frac{\pi}{6}n - 0\right)$
- $\cos\left(\frac{\pi}{6}n - \frac{\pi}{4}\right)$
- $\cos\left(\frac{\pi}{6}n - \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{6}n\right)$
- $\cos\left(\frac{\pi}{6}n - 2\pi\right) = \cos\left(\frac{\pi}{6}n\right)$



Summary

- Sinusoids play a starring role in both the theory and applications of signals and systems
- A sinusoid has a **frequency** and a **phase**
- A complex sinusoid is a helix in three-dimensional space and naturally induces the sine and cosine
- Negative frequency is nothing to be scared by; it just means that the helix spins backwards



Discrete-Time Sinusoids Are Weird

Discrete-Time Sinusoids are Weird!

- Discrete-time sinusoids $e^{j(\omega n+\phi)}$ have two counterintuitive properties
- Both involve the frequency ω
- Weird property #1: Aliasing
- Weird property #2: Aperiodicity

Weird Property #1: Aliasing of Sinusoids

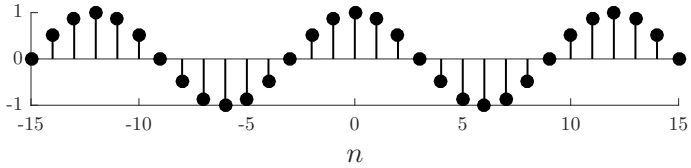
- Consider two sinusoids with two different frequencies
  - $\omega \quad \Rightarrow \quad x_1[n] = e^{j(\omega n + \phi)}$
  - $\omega + 2\pi \quad \Rightarrow \quad x_2[n] = e^{j((\omega+2\pi)n+\phi)}$
- But note that

$$x_2[n] = e^{j((\omega+2\pi)n+\phi)} = e^{j(\omega n+\phi)+j2\pi n} = e^{j(\omega n+\phi)}\, e^{j2\pi n} = e^{j(\omega n+\phi)} = x_1[n]$$

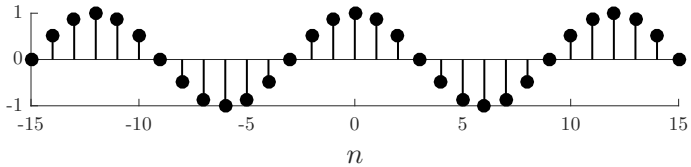
- The signals x_1 and x_2 have different frequencies but are **identical**!
- We say that x_1 and x_2 are aliases; this phenomenon is called **aliasing**
- Note: Any integer multiple of 2π will do; try with $x_3[n] = e^{j((\omega+2\pi m)n+\phi)}$, $m \in \mathbb{Z}$

Aliasing of Sinusoids – Example

- $x_1[n] = \cos\left(\frac{\pi}{6}n\right)$



- $x_2[n] = \cos\left(\frac{13\pi}{6}n\right) = \cos\left((\frac{\pi}{6} + 2\pi)n\right)$



Alias-Free Frequencies

- Since

$$x_3[n] = e^{j(\omega+2\pi m)n+\phi)} = e^{j(\omega n+\phi)} = x_1[n] \quad \forall m \in \mathbb{Z}$$

  the only frequencies that lead to unique (distinct) sinusoids lie in an interval of length 2π

- Convenient to interpret the frequency ω as an **angle**
  (then aliasing is handled automatically; more on this later)

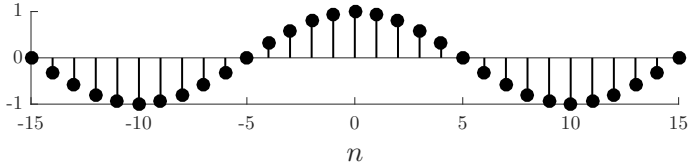
- Two intervals are typically used in the signal processing literature (and in this course)
  - $0 \leq \omega < 2\pi$
  - $-\pi < \omega \leq \pi$

Low and High Frequencies

$$e^{j(\omega n+\phi)}$$

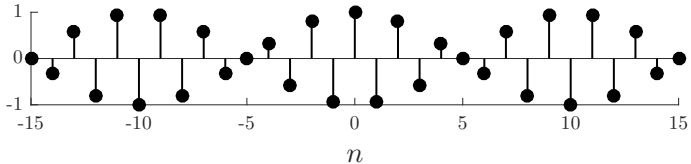
- **Low frequencies:** ω close to 0 or 2π rad

  Ex: $\cos\left(\frac{\pi}{10}n\right)$



- **High frequencies:** ω close to π or $-\pi$ rad

  Ex: $\cos\left(\frac{9\pi}{10}n\right)$



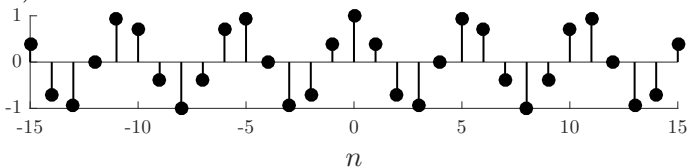
Weird Property #2: Periodicity of Sinusoids

- Consider $x_1[n] = e^{j(\omega n + \phi)}$ with frequency $\omega = \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$ (harmonic frequency)

- It is easy to show that x_1 is periodic with period N, since

$$x_1[n+N] = e^{j(\omega(n+N)+\phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n+\phi)}\, e^{j(\omega N)} = e^{j(\omega n+\phi)}\, e^{j(\frac{2\pi k}{N} N)} = x_1[n] \checkmark$$

- Ex: $x_1[n] = \cos(\frac{2\pi 3}{16} n), \quad N = 16$



- Note: x_1 is periodic with the (smaller) period of $\frac{N}{k}$ when $\frac{N}{k}$ is an integer

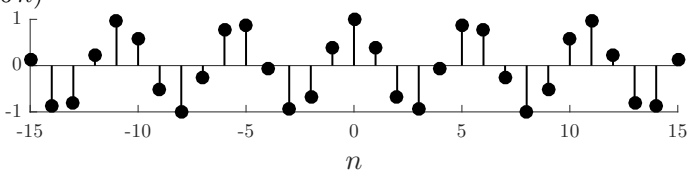
Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n+\phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$ (not harmonic frequency)

- Is x_2 periodic?

$$x_2[n+N] = e^{j(\omega(n+N)+\phi)} = e^{j(\omega n+\omega N+\phi)} = e^{j(\omega n+\phi)}\, e^{j(\omega N)} \neq x_1[n] \quad \text{NO!}$$

- Ex: $x_2[n] = \cos(1.16\, n)$



- If its frequency ω is not harmonic, then a sinusoid oscillates but is not periodic!

Harmonic Sinusoids

$$e^{j(\omega n+\phi)}$$

- Semi-amazing fact: The **only** periodic discrete-time sinusoids are those with **harmonic frequencies**

$$\omega = \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$$

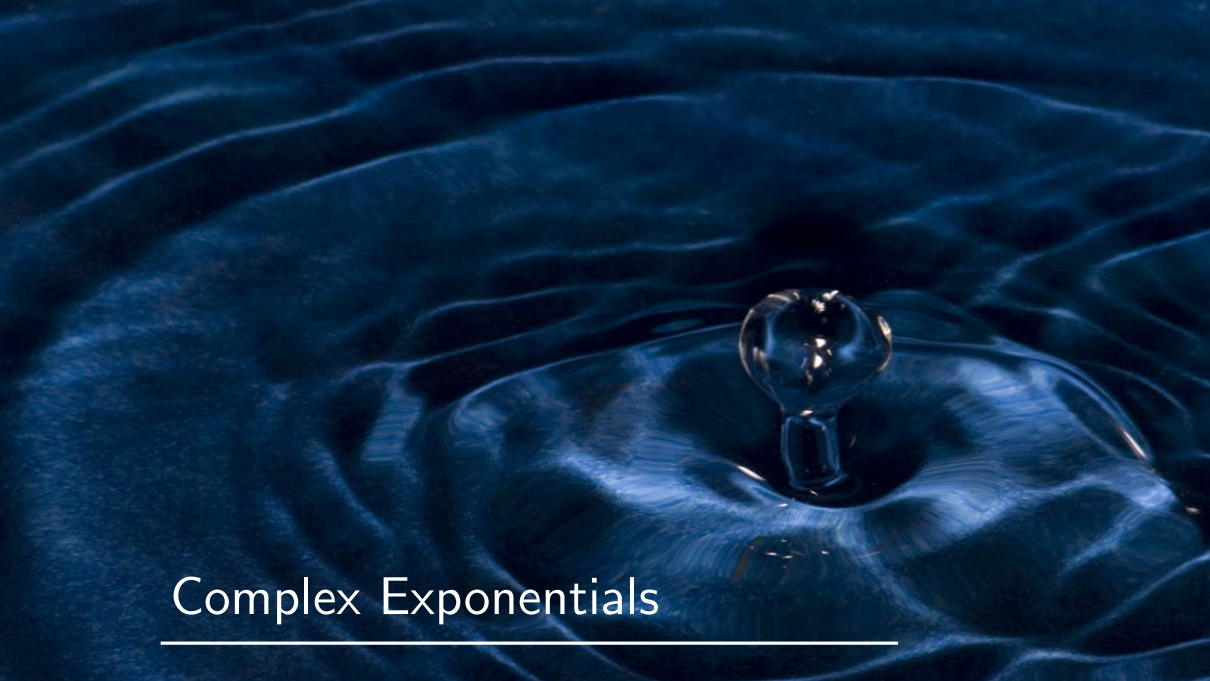
- Which means that
  - **Most** discrete-time sinusoids are **not** periodic!
  - The harmonic sinusoids are somehow magical (they play a starring role later in the DFT)

Harmonic Sinusoids (Matlab)

- **Click here** to view a MATLAB demo that visualizes harmonic sinusoids.

Summary

- Discrete-time sinusoids $e^{j(\omega n+\phi)}$ have two counterintuitive properties
- Both involve the frequency ω
- Weird property #1: Aliasing
- Weird property #2: Aperidiocity
- The only sinusoids that are periodic: Harmonic sinusoids $e^{j(\frac{2\pi k}{N}n+\phi)}$, $n, k, N \in \mathbb{Z}$



Complex Exponentials

Complex Exponential

- Complex sinusoid $e^{j(\omega n+\phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$
- Generalize to $e^{\text{General Complex Numbers}}$
- Consider the general complex number $z = |z|\, e^{j\omega}$, $z \in \mathbb{C}$
  - $|z|$ = magnitude of z
  - $\omega = \angle(z)$, phase angle of z
  - Can visualize $z \in \mathbb{C}$ as a **point** in the **complex plane**
- Now we have

$$z^n = (|z|e^{j\omega})^n = |z|^n (e^{j\omega})^n = |z|^n e^{j\omega n}$$

  - $|z|^n$ is a **real exponential** (a^n with $a = |z|$)
  - $e^{j\omega n}$ is a **complex sinusoid**

Complex Exponential is a Spiral

$$z^n \;=\; \left(|z|\, e^{j\omega}\right)^n \;=\; |z|^n\, e^{j\omega n}$$

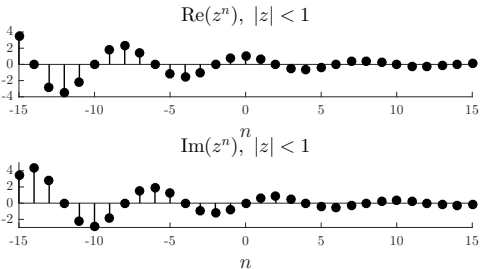
- $|z|^n$ is a **real exponential** envelope $(a^n$ with $a = |z|)$
- $e^{j\omega n}$ is a **complex sinusoid**
- z^n is a helix with expanding radius (spiral)

Complex Exponential is a Spiral

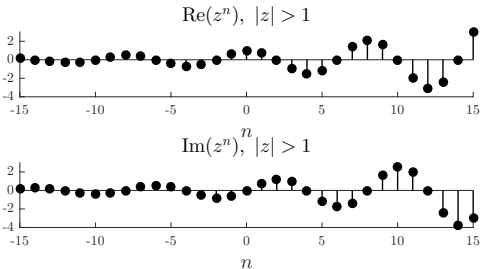
$$z^n \;=\; \left(|z|\, e^{j\omega n}\right)^n \;=\; |z|^n\, e^{j\omega n}$$

- $|z|^n$ is a **real exponential** envelope (a^n with $a = |z|$)
- $e^{j\omega n}$ is a **complex sinusoid**

$|z| < 1$



$|z| > 1$



Complex Exponentials and z Plane (Matlab)

- **Click here** to view a MATLAB demo plotting the signals z^n.

Summary

- Complex sinusoid $e^{j(\omega n+\phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$
- Complex exponential: Generalize $e^{j(\omega n+\phi)}$ to $e^{\text{General Complex Numbers}}$
- A complex exponential is the product of a real exponential and a complex sinusoid
- A complex exponential is a spiral in three-dimensional space

Signals are Vectors

Set B

Signals are Vectors

Signals are Vectors

- Signals are mathematical objects
- Here we will develop tools to analyze the **geometry** of sets of signals
- The tools come from **linear algebra**
- By interpreting signals as vectors in a vector space, we will be able to speak about the length of a signal (its "strength," more below), angles between signals (their similarity), and more
- We will also be able to use matrices to better understand how signal processing systems work
- Caveat: This is not a course on linear algebra!

Vector Space

DEFINITION

A linear **vector space** V is a collection of vectors such that if $x, y \in V$ and α is a scalar then

$$\alpha\, x \in V \qquad \text{and} \qquad x + y \in V$$

- In words:
  - A rescaled vector stays in the vector space
  - The sum of two vectors stays in the vector space
- We will be interested in scalars (basically, numbers) α that either live in $\mathbb{R}$ or $\mathbb{C}$
- Classical vector spaces that you know and love
  - $\mathbb{R}^N$, the set of all vectors of length N with real-valued entries
  - $\mathbb{C}^N$, the set of all vectors of length N with complex-valued entries
  - Special case that we will use all the time to draw pictures and build intuition: $\mathbb{R}^2$

The Vector Space $\mathbb{R}^2$ (1)

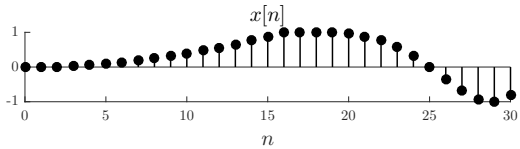
- Vectors in $\mathbb{R}^2$: $x = \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$, $y = \begin{bmatrix} y[0] \\ y[1] \end{bmatrix}$, $x[0], x[1], y[0], y[1] \in \mathbb{R}$
  - Note: We will enumerate the entries of a vector starting from 0 rather than 1 (this is the convention in signal processing and programming languages like "C", but not in Matlab)
  - Note: We will not use the traditional boldface or underline notation for vectors
- Scalars: $\alpha \in \mathbb{R}$
- Scaling: $\alpha\, x = \alpha \begin{bmatrix} x[0] \\ x[1] \end{bmatrix} = \begin{bmatrix} \alpha\, x[0] \\ \alpha\, x[1] \end{bmatrix}$

The Vector Space $\mathbb{R}^2$ (2)

- Vectors in $\mathbb{R}^2$: $x = \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$, $y = \begin{bmatrix} y[0] \\ y[1] \end{bmatrix}$, $x[0], x[1], y[0], y[1] \in \mathbb{R}$
- Scalars: $\alpha \in \mathbb{R}$
- Summing: $x + y = \begin{bmatrix} x[0] \\ x[1] \end{bmatrix} + \begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} x[0] + y[0] \\ x[1] + y[1] \end{bmatrix}$

The Vector Space $\mathbb{R}^N$

- Vectors in $\mathbb{R}^N$: $x = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$, $x[n] \in \mathbb{R}$

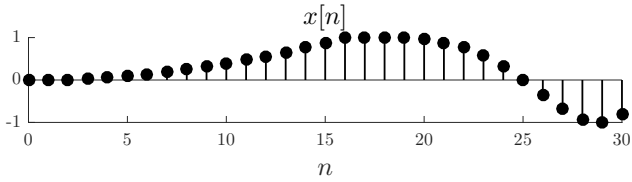


- This is exactly the same as a real-valued discrete time signal; that is, **signals are vectors**
  - Scaling αx amplifies/attenuates a signal by the factor α
  - Summing $x + y$ creates a new signal that mixes x and y

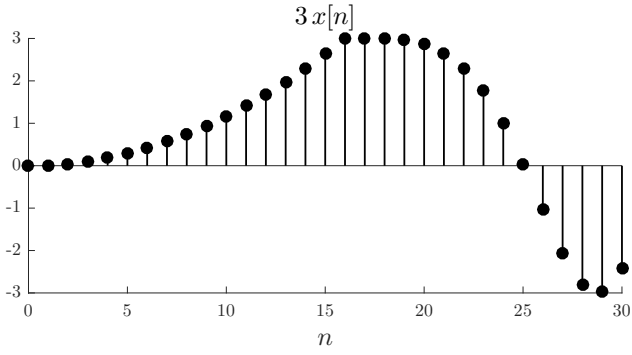
- $\mathbb{R}^N$ is harder to visualize than $\mathbb{R}^2$ and $\mathbb{R}^3$, but the intuition gained from $\mathbb{R}^2$ and $\mathbb{R}^3$ generally holds true with no surprises (at least in this course)

The Vector Space $\mathbb{R}^N$ – Scaling

- Signal $x[n]$

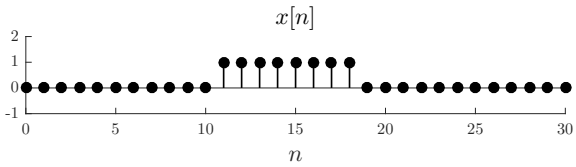


- Scaled signal $3\,x[n]$

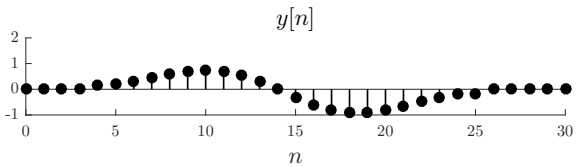


The Vector Space $\mathbb{R}^N$ – Summing

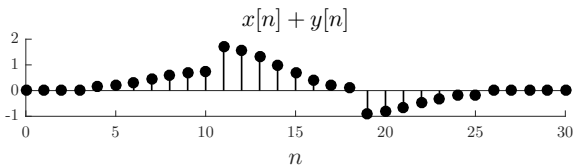
- Signal $x[n]$



- Signal $y[n]$



- Sum $x[n] + y[n]$



The Vector Space $\mathbb{C}^N$ (1)

- $\mathbb{C}^N$ is the same as $\mathbb{R}^N$ with a few minor modifications
- Vectors in $\mathbb{C}^N$: $x = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$, $x[n] \in \mathbb{C}$
- Each entry $x[n]$ is a complex number that can be represented as

$$x[n] = \mathrm{Re}\{x[n]\} + j\,\mathrm{Im}\{x[n]\} = |x[n]|\, e^{j\angle x[n]}$$

- Scalars $\alpha \in \mathbb{C}$

The Vector Space $\mathbb{C}^N$ (2)

- Rectangular form

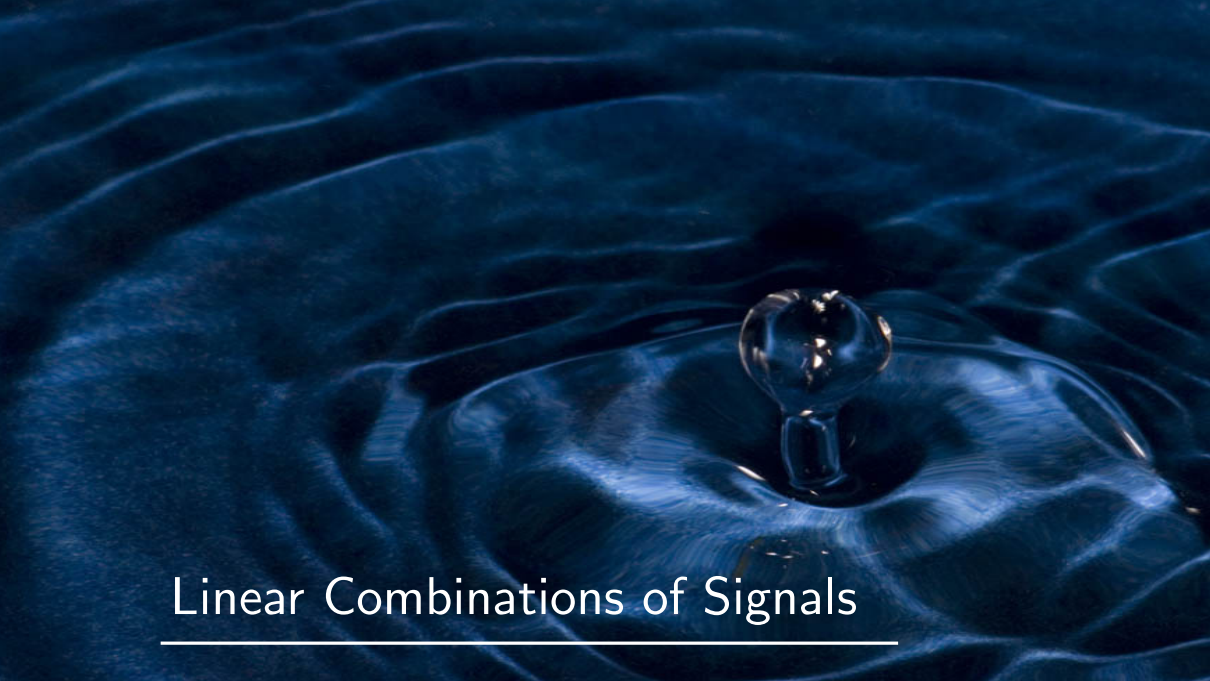
$$x = \begin{bmatrix} \mathrm{Re}\{x[0]\} + j\,\mathrm{Im}\{x[0]\} \\ \mathrm{Re}\{x[1]\} + j\,\mathrm{Im}\{x[1]\} \\ \vdots \\ \mathrm{Re}\{x[N-1]\} + j\,\mathrm{Im}\{x[N-1]\} \end{bmatrix} = \mathrm{Re}\left\{ \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \right\} + j\,\mathrm{Im}\left\{ \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \right\}$$

- Polar form

$$x = \begin{bmatrix} |x[0]|\, e^{j\angle x[0]} \\ |x[1]|\, e^{j\angle x[1]} \\ \vdots \\ |x[N-1]|\, e^{j\angle x[N-1]} \end{bmatrix}$$

Summary

- Linear algebra provides powerful tools to study signals and systems
- Signals are **vectors** that live in a vector space
- Of particular interest in signal processing are the vector spaces $\mathbb{R}^N$ and $\mathbb{C}^N$



Linear Combinations of Signals

Linear Combination of Signals

- Many signal processing applications feature **sums** of a number of signals

DEFINITION

Given a collection of M vectors $x_0, x_1, \dots x_{M-1} \in \mathbb{C}^N$ and M scalars $\alpha_0, \alpha_1, \dots, \alpha_{M-1} \in \mathbb{C}$, the **linear combination** of the vectors is given by

$$y = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_{M-1} x_{M-1} = \sum_{m=0}^{M-1} \alpha_m x_m$$

- Clearly the result of the linear combination is a vector $y \in \mathbb{C}^N$

Linear Combination Example

- A recording studio uses a **mixing board** (or desk) to create a linear combination of the signals from the different instruments that make up a song
- Say $x_0 =$ drums, $x_1 =$ bass, $x_2 =$ guitar, $\ldots$, $x_{22} =$ saxophone, $x_{23} =$ singer ($M = 24$)
- Linear combination (output of mixing board)

$$y = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_{23} x_{23} = \sum_{m=0}^{23} \alpha_m x_m$$

- Changing the α_m's results in a different "mix" y that emphasizes/deemphasizes certain instruments

Linear Combination = Matrix Multiplication

- Step 1: Stack the vectors $x_m \in \mathbb{C}^N$ as column vectors into an $N \times M$ matrix

$$X = \left[x_0 | x_1 | \cdots | x_{M-1}\right]$$

- Step 2: Stack the scalars α_m into an $M \times 1$ column vector

$$a = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix}$$

- Step 3: We can now write a linear combination as the matrix/vector product

$$y = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_{M-1} x_{M-1} = \sum_{m=0}^{M-1} \alpha_m x_m = \left[x_0 | x_1 | \cdots | x_{M-1}\right] \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix} = Xa$$

Linear Combination = Matrix Multiplication (The Gory Details)

- M vectors in $\mathbb{C}^N$: $x_m = \begin{bmatrix} x_m[0] \\ x_m[1] \\ \vdots \\ x_m[N-1] \end{bmatrix}$, $m = 0, 1, \ldots, M-1$
- $N \times M$ matrix: $X = \begin{bmatrix} x_0[0] & x_1[0] & \cdots & x_{M-1}[0] \\ x_0[1] & x_1[1] & \cdots & x_{M-1}[1] \\ \vdots & \vdots & & \vdots \\ x_0[N-1] & x_1[N-1] & \cdots & x_{M-1}[N-1] \end{bmatrix}$
- Note: The row-n, column-m element of the matrix $[X]_{n,m} = x_m[n]$
- M scalars α_m, $m = 0, 1, \ldots, M-1$: $a = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix}$
- Linear combination $y = Xa$

Linear Combination = Matrix Multiplication (Summary)

- Linear combination $y = Xa$

- The row-n, column-m element of the $N \times M$ matrix $[X]_{n,m} = x_m[n]$

$$y = \begin{bmatrix} \vdots \\ y[n] \\ \vdots \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & x_m[n] & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ \alpha_m \\ \vdots \end{bmatrix} = Xa$$

- Sum-based formula for $y[n]$

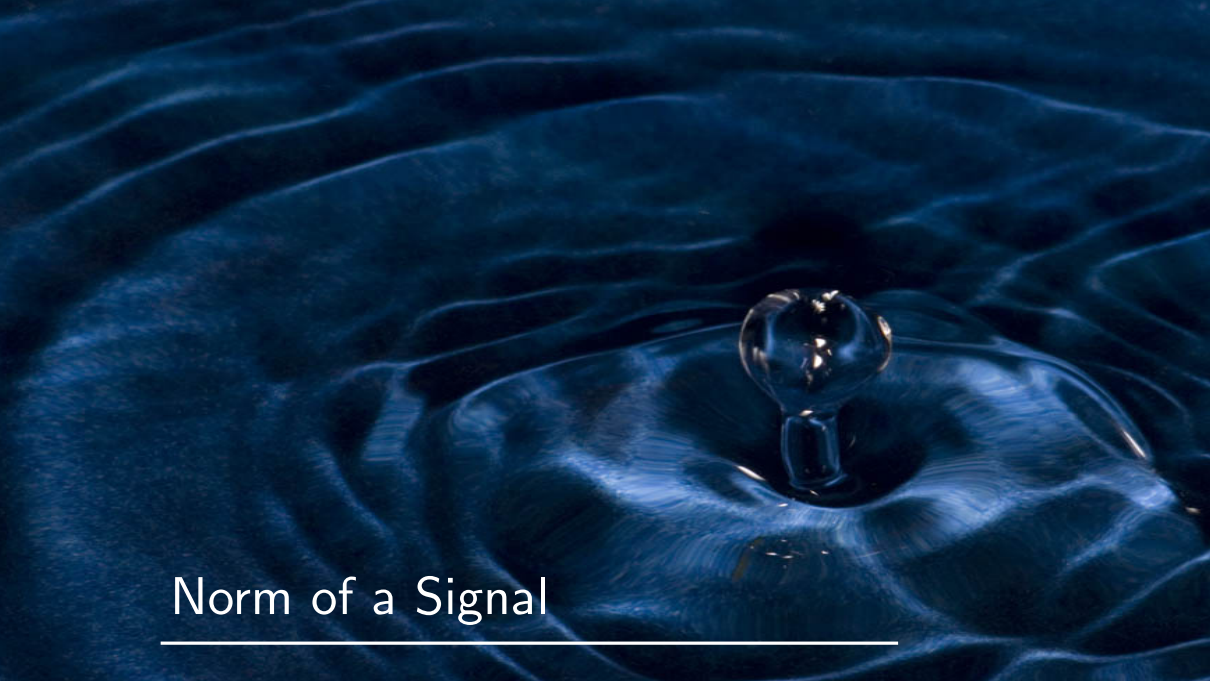
$$y[n] = \sum_{m=0}^{M-1} \alpha_m \, x_m[n]$$

Linear Combination as Matrix Multiplication (Matlab)

- **Click here** to view a MATLAB demo using sound to explore linear combinations.

Summary

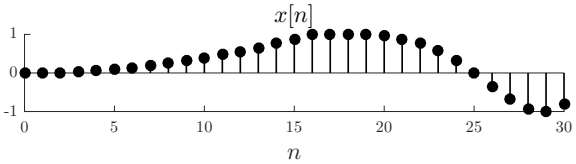
- Linear algebra provides power tools to study signals and systems
- Signals are **vectors** that live in a vector space
- We can combine several signals to form one new signal via a **linear combination**
- Linear combination is basically a matrix/vector multiplication



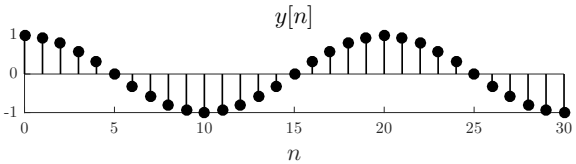
Norm of a Signal

Strength of a Vector

- How to quantify the "strength" of a vector?
- How to say that one signal is "stronger" than another?
- Signal x



- Signal y



Strength of a Vector: 2-Norm

DEFINITION

The **Euclidean length**, or 2-**norm**, of a vector $x \in \mathbb{C}^N$ is given by

$$\|x\|_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2}$$

The **energy** of x is given by $(\|x\|_2)^2 = \|x\|_2^2$

- The norm takes as input a vector in $\mathbb{C}^N$ and produces a **real number** that is ≥ 0
- When it is clear from context, we will suppress the subscript "2" in $\|x\|_2$ and just write $\|x\|$

2-Norm Example

- Ex: $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- ℓ_2 norm

$$\|x\|_2 \;=\; \sqrt{\sum_{n=0}^{N-1} |x[n]|^2} \;=\; \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

Strength of a Vector: p-Norm

- The Euclidean length is not the only measure of "strength" of a vector in $\mathbb{C}^N$

DEFINITION

The p-**norm** of a vector $x \in \mathbb{C}^N$ is given by

$$\|x\|_p = \left(\sum_{n=0}^{N-1} |x[n]|^p \right)^{1/p}$$

DEFINITION

The 1-**norm** of a vector $x \in \mathbb{C}^N$ is given by

$$\|x\|_1 = \sum_{n=0}^{N-1} |x[n]|$$

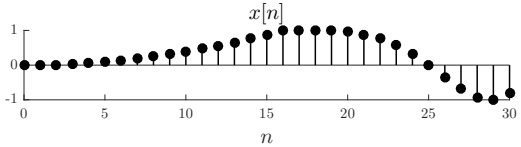
Strength of a Vector: ∞-Norm

DEFINITION

The ∞-**norm** of a vector $x \in \mathbb{C}^N$ is given by

$$\|x\|_\infty = \max_n |x[n]|$$

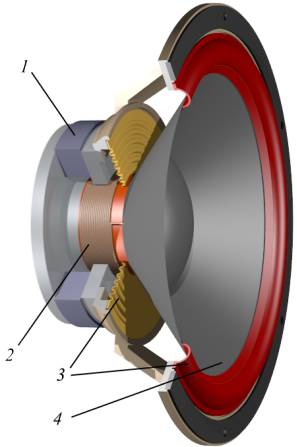
- $\|x\|_\infty$ is simply the largest entry in the vector x (in absolute value)



- While $\|x\|_2^2$ measures the **energy** in a signal, $\|x\|_\infty$ measures the **peak** value (of the magnitude); both are very useful in applications
- Interesting mathematical fact: $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$

Physical Significance of Norms (1)

- Two norms have special physical significance
  - $\|x\|_2^2$: energy in x
  - $\|x\|_\infty$: peak value in x
- A **loudspeaker** is a transducer that converts electrical signals into acoustic signals
- Conventional loudspeakers consist of a paper cone (4) that is joined to a coil of wire (2) that is wound around a permanent magnet (1)
- If the energy $\|x\|_2^2$ is too large, then the coil of wire will melt from excessive heating
- If the peak value $\|x\|_\infty$ is too large, then the large back and forth excursion of the coil of wire will tear it off of the paper cone



Physical Significance of Norms (2)

- Consider a **robotic car** that we wish to guide down a roadway
- How to measure the amount of deviation from the center of the driving lane?
- Let x be a vector of measurements of the car's GPS position and let y be a vector containing the GPS positions of the center of the driving lane
- Clearly we would like to make the error signal $y - x$ "small"; but how to measure smallness?
- Minimizing $\|y - x\|_2^2$, energy in the error signal, will tolerate a few large deviations from the lane center (not very safe)
- Minimizing $\|y - x\|_\infty$, the maximum of the error signal, will not tolerate any large deviations from the lane center (much safer)

Normalizing a Vector

DEFINITION

A vector x is **normalized** (in the 2-norm) if $\|x\|_2 = 1$

- Normalizing a vector is easy; just scale it by $\frac{1}{\|x\|_2}$
- Ex: $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $||x||_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

$$x' = \frac{1}{\sqrt{14}} x = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}, \qquad ||x'||_2 = 1$$

Summary

- Linear algebra provides power tools to study signals and systems
- Signals are **vectors** that live in a vector space
- Norms measure the "strength" of a signal; we introduced the 2- 1-, and ∞-norms

Inner Product

The Geometry of Signals

- Up to this point, we have developed the viewpoint of "signals as vectors" in a vector space
- We have focused on quantities related to individual vectors, ex: norm (strength)
- Now we turn to a quantity related to pairs of vectors, the **inner product**
- A powerful and ubiquitous signal processing tool

Aside: Transpose of a Vector

- Recall that the **transpose** operation ^{T} converts a column vector to a row vector (and vice versa)

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}^T = \begin{bmatrix} x[0] & x[1] & \cdots & x[N-1] \end{bmatrix}$$

- In addition to transposition, the **conjugate transpose** (aka Hermitian transpose) operation ^{H} takes the complex conjugate

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}^H = \begin{bmatrix} x[0]^* & x[1]^* & \cdots & x[N-1]^* \end{bmatrix}$$

Inner Product

DEFINITION

The **inner product** (or dot product) between two vectors $x, y \in \mathbb{C}^N$ is given by

$$\langle x, y \rangle = y^H x = \sum_{n=0}^{N-1} x[n]\, y[n]^*$$

- The inner product takes two signals (vectors in $\mathbb{C}^N$) and produces a single (complex) number
- **Angle** between two vectors $x, y \in \mathbb{R}^N$

$$\cos\theta_{x,y} = \frac{\langle x, y \rangle}{\|x\|_2 \, \|y\|_2}$$

- **Angle** between two vectors $x, y \in \mathbb{C}^N$

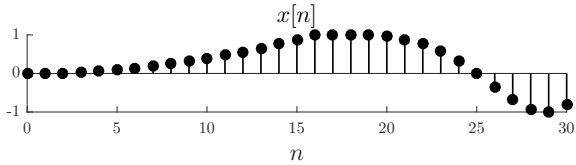
$$\cos\theta_{x,y} = \frac{\mathrm{Re}\{\langle x, y \rangle\}}{\|x\|_2 \, \|y\|_2}$$

Inner Product Example 1

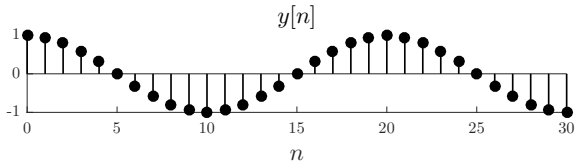
- Consider two vectors in $\mathbb{R}^2$: $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- $\|x\|_2^2 = 1^2 + 2^2 = 5, \qquad \|y\|_2^2 = 3^2 + 2^2 = 13$
- $\theta_{x,y} = \arccos\left(\frac{1\times 3 \,+\, 2\times 2}{\sqrt{5}\sqrt{13}}\right) = \arccos\left(\frac{7}{\sqrt{65}}\right) \approx 0.519 \text{ rad} \approx 29.7^\circ$

Inner Product Example 2

- Signal x



- Signal y



- Inner product computed using Matlab: $\langle x, y \rangle = y^T x = 5.995$
- Angle computed using Matlab: $\theta_{x,y} = 64.9^\circ$

2-Norm from Inner Product

- Question: What's the inner product of a signal with itself?

$$\langle x, x \rangle = \sum_{n=0}^{N-1} x[n]\, x[n]^* = \sum_{n=0}^{N-1} |x[n]|^2 = \|x\|_2^2$$

- Answer: The 2-norm!

- Mathematical aside: This property makes the 2-norm very special; no other p-norm can be computed via the inner product like this

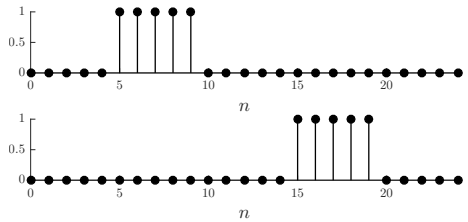
Orthogonal Vectors

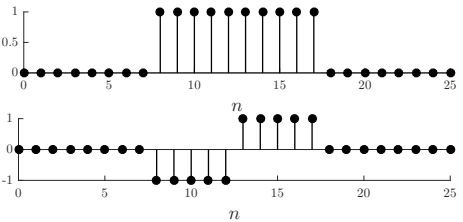
DEFINITION

Two vectors $x, y \in \mathbb{C}^N$ are **orthogonal** if

$$\langle x, y \rangle = 0$$

- $\langle x, y \rangle = 0 \;\Rightarrow\; \theta_{x,y} = \frac{\pi}{2} \text{ rad} = 90^\circ$
- Ex: Two sets of orthogonal signals





Harmonic Sinusoids are Orthogonal

$$s_k[n] = e^{j\frac{2\pi k}{N}n}, \quad n, k, N \in \mathbb{Z}, \quad 0 \le n \le N-1, \quad 0 \le k \le N-1$$

- Claim: $\langle s_k, s_l \rangle = 0, \quad k \neq l$ (a key result for the DFT)
- Verify by direct calculation

$$\begin{aligned}
\langle s_k, s_l \rangle &= \sum_{n=0}^{N-1} s_k[n]\, s_l^*[n] = \sum_{n=0}^{N-1} e^{j\frac{2\pi k}{N}n} \left(e^{j\frac{2\pi l}{N}n}\right)^* = \sum_{n=0}^{N-1} e^{j\frac{2\pi k}{N}n}\, e^{-j\frac{2\pi l}{N}n} \\
&= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)n} \quad \text{let } r = k-l \in \mathbb{Z}, r \neq 0 \\
&= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}rn} = \sum_{n=0}^{N-1} a^n \quad \text{with } a = e^{j\frac{2\pi}{N}r}, \text{then use } \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \\
&= \frac{1 - e^{j\frac{2\pi r N}{N}}}{1 - e^{j\frac{2\pi r}{N}}} = 0 \ \checkmark
\end{aligned}$$

Harmonic Sinusoids are Orthogonal (Matlab)

- **Click here** to view a MATLAB demo exploring the orthogonality of harmonic sinusoids.

Normalizing Harmonic Sinusoids

$$s_k[n] = e^{j\frac{2\pi k}{N}n}, \quad n, k, N \in \mathbb{Z}, \;\; 0 \le n \le N-1, \;\; 0 \le k \le N-1$$

- Claim: $\|s_k\|_2 = \sqrt{N}$
- Verify by direct calculation

$$\|s_k\|_2^2 \;=\; \sum_{n=0}^{N-1} |s_k[n]|^2 \;=\; \sum_{n=0}^{N-1} |e^{j\frac{2\pi k}{N}n}|^2 \;=\; \sum_{n=0}^{N-1} 1 \;=\; N \quad \checkmark$$

- **Normalized harmonic sinusoids**

$$\widetilde{s}_k[n] = \frac{1}{\sqrt{N}}\, e^{j\frac{2\pi k}{N}n}, \quad n, k, N \in \mathbb{Z}, \;\; 0 \le n \le N-1, \;\; 0 \le k \le N-1$$

Summary

- **Inner product** measures the similarity between two signals

$$\langle x, y \rangle = y^H x = \sum_{n=0}^{N-1} x[n]\, y[n]^*$$

- Angle between two signals

$$\cos \theta_{x,y} = \frac{\mathrm{Re}\{\langle x, y \rangle\}}{\|x\|_2 \, \|y\|_2}$$

- Harmonic sinusoids are **orthogonal** (as well as periodic)

Matrix Multiplication and Inner Product

Recall: Matrix Multiplication as a Linear Combination of Columns

- Consider the (real- or complex-valued) matrix multiplication $y = Xa$
- The row-n, column-m element of the $N \times M$ matrix $[X]_{n,m} = x_m[n]$
- We can compute y as a **linear combination** of the **columns of** X weighted by the elements in a

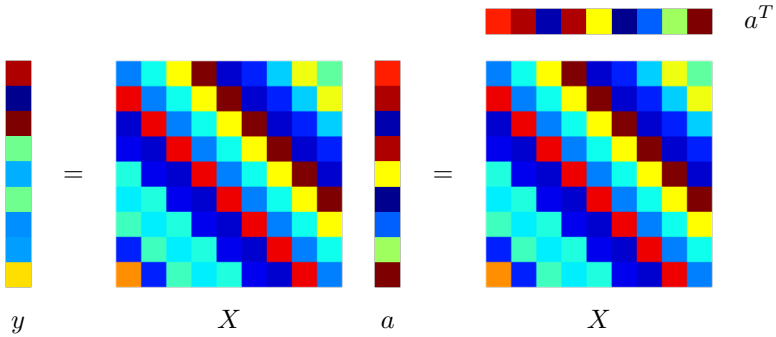
$$y = \begin{bmatrix} \vdots \\ y[n] \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ x_0[n] & x_1[n] & \cdots & x_{M-1}[n] \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix} = Xa$$

- Sum-based formula for $y[n]$

$$y[n] = \sum_{m=0}^{M-1} \alpha_m \, x_m[n], \; = \sum_{m=0}^{M-1} \alpha_m \, (\text{column } m \text{ of } X), \quad 0 \le n \le N-1$$

Visualizing Matrix Multiplication as a Linear Combination of Columns

- Consider the (real- or complex-valued) matrix multiplication $y = Xa$
- We can compute y as a **linear combination** of the **columns of** X weighted by the elements in a



Matrix Multiplication as a Sequence of Inner Products of Rows

- Consider the **real-valued** matrix multiplication $y = Xa$
- The row-n, column-m element of the $N \times M$ matrix $[X]_{n,m} = x_m[n]$
- We can compute each element $y[n]$ in y as the **inner product** of the **n-th row of X** with the vector a

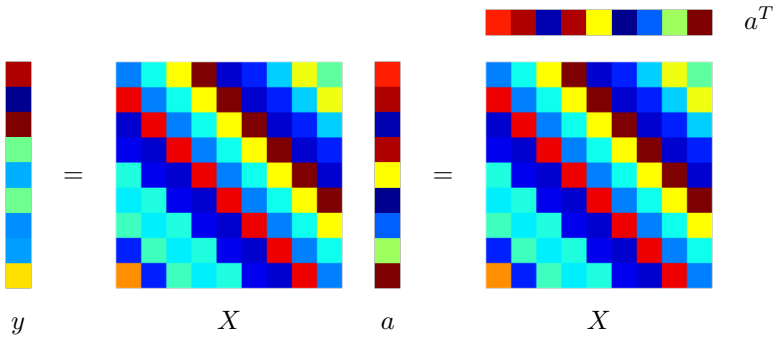
$$y = \begin{bmatrix} \vdots \\ y[n] \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ x_0[n] & x_1[n] & \cdots & x_{M-1}[n] \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix} = Xa$$

- Can write $y[n]$

$$y[n] = \sum_{m=0}^{M-1} \alpha_m \, x_m[n] = \langle (\text{row } n \text{ of } X)^T, \, a \rangle, \quad 0 \le n \le N-1$$

Visualizing Matrix Multiplication as a Sequence of Inner Products of Rows

- Consider the **real-valued** matrix multiplication $y = Xa$
- We can compute each element $y[n]$ in y as the **inner product** of the n-**th row of** X with the vector a



Matrix Multiplication as a Sequence of Inner Products of Rows

- What about **complex-valued** matrix multiplication $y = Xa$?
- The same interpretation works, but we need to use the following "**inner product**"

$$y[n] \;=\; \sum_{m=0}^{M-1} \alpha_m \, x_m[n] \;\neq\; \langle (\text{row } n \text{ of } X)^T,\, a \rangle, \quad 0 \leq n \leq N-1$$

- Note: This is *nearly* the inner product for complex signals except that is lacking the complex conjugation
- We will often abuse notation by calling this an inner product

Summary

- Given the matrix/vector product $y = Xa$, we can compute each element $y[n]$ in y as the **inner product** of the n-**th row of** X with the vector a

- Not strictly true for complex matrices/vectors, but the interpretation is useful nevertheless

Cauchy Schwarz Inequality

Comparing Signals

- Inner product and angle between vectors enable us to **compare signals**

$$\langle x, y \rangle = y^H x = \sum_{n=0}^{N-1} x[n]\, y[n]^*$$

$$\cos \theta_{x,y} = \frac{\mathrm{Re}\{\langle x, y \rangle\}}{\|x\|_2 \, \|y\|_2}$$

- The Cauchy Schwarz Inequality quantifies the comparison
- A powerful and ubiquitous signal processing tool
- Note: Our development will emphasize intuition over rigor

Cauchy-Schwarz Inequality (1)

- Focus on real-valued signals in $\mathbb{R}^N$ (the extension to $\mathbb{C}^N$ is easy)
- Recall that $\cos\theta_{x,y} = \frac{\langle x,y\rangle}{\|x\|_2\,\|y\|_2}$
- Now, use the fact that $0 \leq |\cos\theta| \leq 1$ to write

$$0 \;\leq\; \left|\frac{\langle x,y\rangle}{\|x\|_2\,\|y\|_2}\right| \;\leq\; 1$$

- Rewrite as the **Cauchy-Schwarz Inequality** (CSI)

$$0 \;\leq\; |\langle x,y\rangle| \;\leq\; \|x\|_2\,\|y\|_2$$

- Interpretation: The inner product $\langle x,y\rangle$ measures the **similarity** of x to y

Cauchy-Schwarz Inequality (2)

$$0 \;\leq\; |\langle x, y\rangle| \;\leq\; \|x\|_2 \, \|y\|_2$$

- Interpretation: The inner product $\langle x, y\rangle$ measures the **similarity** of x to y
- Two extreme cases:
  - Lower bound: $\langle x, y\rangle = 0$ or $\theta_{x,y} = 90°$: x and y are most different when they are orthogonal
  - Upper bound: $\langle x, y\rangle = \|x\|_2 \|y\|_2$ or $\theta_{x,y} = 0°$: x and y are most similar when they are collinear (aka linearly dependent, $y = \alpha x$)
- It is hard to understate the importance and ubiquity of the CSI!

Cauchy-Schwarz Inequality Applications

- How does a digital communication system decide whether the signal corresponding to a "0" was transmitted or the signal corresponding to a "1"? (Hint: CSI)
- How does a radar or sonar system find targets in the signal it receives after transmitting a pulse? (Hint: CSI)
- How do many computer vision systems find faces in images? (Hint: CSI)

Cauchy-Schwarz Inequality (Matlab)

- **Click here** to view a MATLAB demo illustrating the Cauchy-Schwarz inequality.

Summary

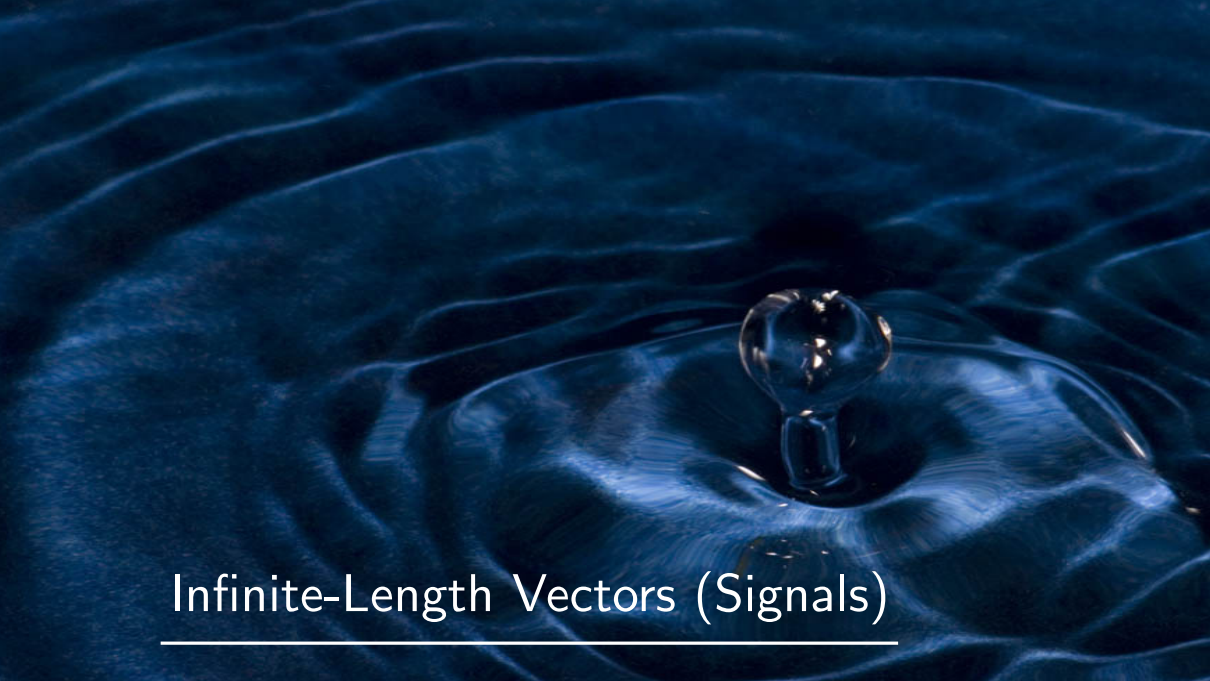
- **Inner product** measures the similarity between two signals

$$\langle x, y \rangle = y^H x = \sum_{n=0}^{N-1} x[n]\, y[n]^*$$

- **Cauchy-Schwarz Inequality** (CSI) calibrates the inner product

$$0 \leq \left| \frac{\langle x, y \rangle}{\|x\|_2 \, \|y\|_2} \right| \leq 1$$

  - Similar signals – close to upper bound (1)
  - Different signals – close to lower bound (0)

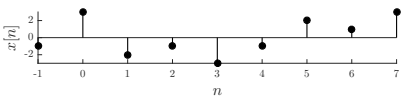


Infinite-Length Vectors (Signals)

From Finite to Infinite-Length Vectors

- Up to this point, we have developed some useful tools for dealing with finite-length vectors (signals) that live in $\mathbb{R}^N$ or $\mathbb{C}^N$: Norms, Inner product, Linear combination
- It turns out that these tools can be generalized to infinite-length vectors (sequences) by letting $N \to \infty$ (infinite-dimensional vector space, aka Hilbert Space)

$$x[n], \quad -\infty < n < \infty, \qquad x = \begin{bmatrix} \vdots \\ x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \\ \vdots \end{bmatrix}$$



- Obviously such a signal cannot be loaded into Matlab; however this viewpoint is still useful in many situations
- We will spell out the generalizations with emphasis on what changes from the finite-length case

2-Norm of an Infinite-Length Vector

DEFINITION

The 2-**norm** of an infinite-length vector x is given by

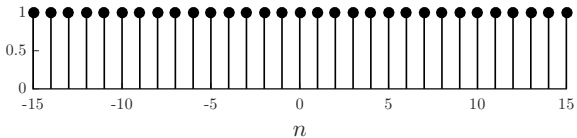
$$\|x\|_2 = \sqrt{\sum_{n=-\infty}^{\infty} |x[n]|^2}$$

The **energy** of x is given by $(\|x\|_2)^2 = \|x\|_2^2$

- When it is clear from context, we will suppress the subscript "2" in $\|x\|_2$ and just write $\|x\|$
- What changes from the finite-length case: Not every infinite-length vector has a finite 2-norm

ℓ_2 Norm of an Infinite-Length Vector – Example

- Signal: $x[n] = 1, \quad -\infty < n < \infty$



- 2-norm:

$$\|x\|_2^2 \;=\; \sum_{n=-\infty}^{\infty} |x[n]|^2 \;=\; \sum_{n=-\infty}^{\infty} 1 \;=\; \infty$$

- Infinite energy!

p- and 1-Norms of an Infinite-Length Vector

DEFINITION

The p-**norm** of an infinite-length vector x is given by

$$\|x\|_p = \left(\sum_{n=-\infty}^{\infty} |x[n]|^p \right)^{1/p}$$

DEFINITION

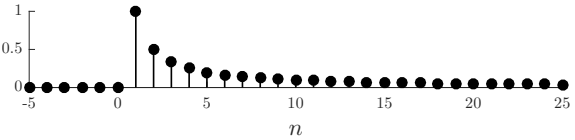
The 1-**norm** of an infinite-length vector x is given by

$$\|x\|_1 = \sum_{n=-\infty}^{\infty} |x[n]|$$

- What changes from the finite-length case: Not every infinite-length vector has a finite p-norm

1- and 2-Norms of an Infinite-Length Vector – Example

- Signal: $x[n] = \begin{cases} 0 & n \leq 0 \\ \frac{1}{n} & n \geq 1 \end{cases}$



- 1-norm

$$\|x\|_1 = \sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

- 2-norm

$$\|x\|_2^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=1}^{\infty} \left|\frac{1}{n}\right|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64 < \infty$$

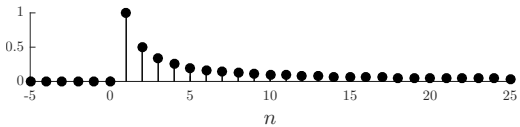
∞-Norm of an Infinite-Length Vector

DEFINITION

The ∞-**norm** of an infinite-length vector x is given by

$$\|x\|_\infty = \sup_n |x[n]|$$

- What changes from the finite-length case: "sup" is a generalization of $\max$ to infinite-length signals that lies beyond the scope of this course
- In both of the above examples, $\|x\|_\infty = 1$



Inner Product of Infinite-Length Signals

DEFINITION

The **inner product** between two infinite-length vectors x, y is given by

$$\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x[n]\, y[n]^*$$

- The inner product takes two signals and produces a **single (complex) number**
- **Angle** between two real-valued signals

$$\cos\theta_{x,y} = \frac{\langle x, y \rangle}{\|x\|_2 \, \|y\|_2}$$

- **Angle** between two complex-valued signals

$$\cos\theta_{x,y} = \frac{\mathrm{Re}\{\langle x, y \rangle\}}{\|x\|_2 \, \|y\|_2}$$

Linear Combination of Infinite-Length Vectors

- The concept of a linear combination extends to infinite-length vectors

- What changes from the finite-length case: We will be especially interested in linear combinations of infinitely many infinite-length vectors

$$y = \sum_{m=-\infty}^{\infty} \alpha_m x_m$$

Linear Combination = Infinite Matrix Multiplication

- Step 1: Stack the vectors x_m as column vectors into a "matrix" with infinitely many rows and columns

$$X = \left[\cdots | x_{-1} | x_0 | x_1 | \cdots\right]$$

- Step 2: Stack the scalars α_m into an infinitely tall column vector $a = \begin{bmatrix} \vdots \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \end{bmatrix}$

- Step 3: We can now write a linear combination as the matrix/vector product

$$y = \sum_{m=-\infty}^{\infty} \alpha_m x_m = \left[\cdots | x_{-1} | x_0 | x_1 | \cdots\right] \begin{bmatrix} \vdots \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \end{bmatrix} = Xa$$

Linear Combination = Infinite Matrix Multiplication (The Gory Details)

- Vectors: $x_m = \begin{bmatrix} \vdots \\ x_m[-1] \\ x_m[0] \\ x_m[1] \\ \vdots \end{bmatrix}$, $-\infty < m < \infty$, and Scalars: $a = \begin{bmatrix} \vdots \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \end{bmatrix}$
- Infinite matrix: $X = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & x_{-1}[-1] & x_0[-1] & x_1[-1] & \cdots \\ \cdots & x_{-1}[0] & x_0[0] & x_1[0] & \cdots \\ \cdots & x_{-1}[1] & x_0[1] & x_1[1] & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$
- Note: The row-n, column-m element of the matrix $[X]_{n,m} = x_m[n]$
- Linear combination $= Xa$

Linear Combination = Infinite Matrix Multiplication (Summary)

- Linear combination $y = Xa$

- The row-n, column-m element of the infinitely large matrix $[X]_{n,m} = x_m[n]$

$$y = \begin{bmatrix} \vdots \\ y[n] \\ \vdots \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & x_m[n] & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ \alpha_m \\ \vdots \end{bmatrix} = Xa$$

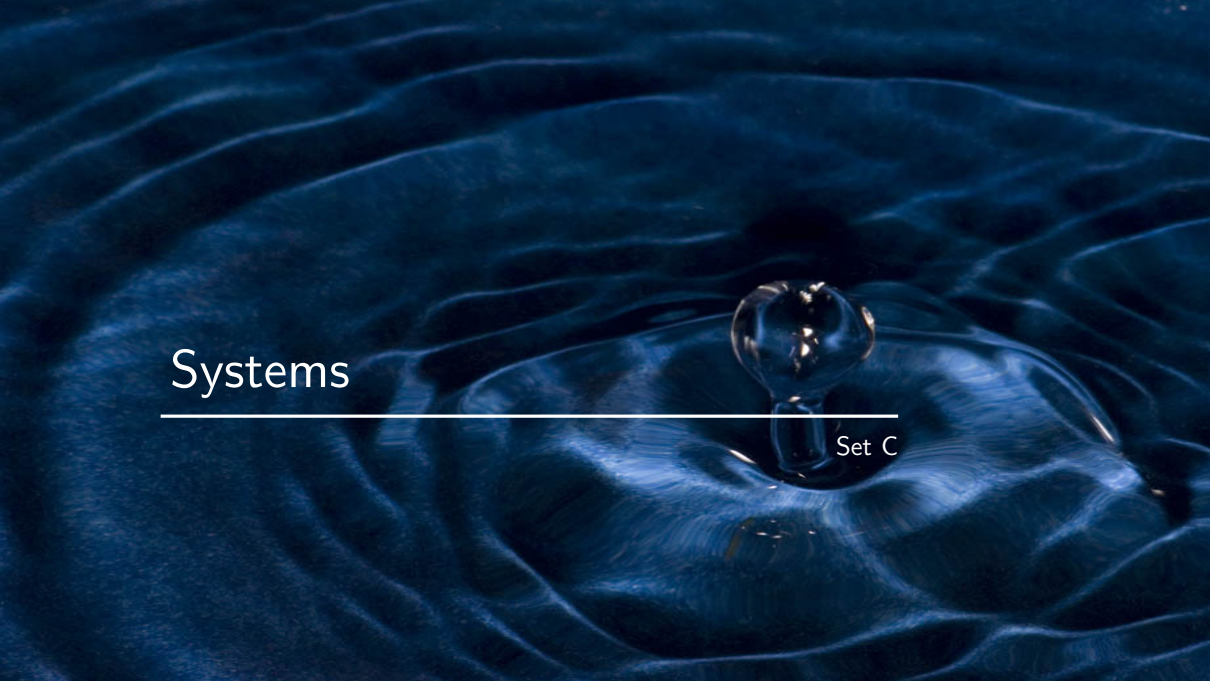
- Sum-based formula for $y[n]$

$$y[n] = \sum_{m=-\infty}^{\infty} \alpha_m \, x_m[n]$$

Summary

- Linear algebra concepts like norm, inner product, and linear combination work apply as well to infinite-length signals as with finite-length signals

- Only a few changes from the finite-length case
  - Not every infinite-length vector has a finite 1-, 2-, or ∞-norm
  - Linear combinations can involve infinitely many vectors



Systems

Set C

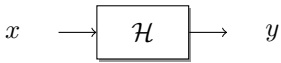
Systems

Systems are Transformations

DEFINITION

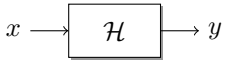
A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$



- Systems manipulate the information in signals
- Examples:
  - A speech recognition system converts acoustic waves of speech into text
  - A radar system transforms the received radar pulse to estimate the position and velocity of targets
  - A functional magnetic resonance imaging (fMRI) system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
  - A 30 day moving average smooths out the day-to-day variability in a stock price

Signal Length and Systems



- Recall that there are two kinds of signals: infinite-length and finite-length
- Accordingly, we will consider two kinds of systems:
  1. Systems that transform an infinite-length-signal x into an infinite-length signal y
  2. Systems that transform a length-N signal x into a length-N signal y

     (Such systems can also be used to process periodic signals with period N)
- For generality, we will assume that the input and output signals are complex valued

System Examples (1)

- Identity

$$y[n] = x[n] \quad \forall n$$

- Scaling

$$y[n] = 2\,x[n] \quad \forall n$$

- Offset

$$y[n] = x[n] + 2 \quad \forall n$$

- Square signal

$$y[n] = (x[n])^2 \quad \forall n$$

- Shift

$$y[n] = x[n+2] \quad \forall n$$

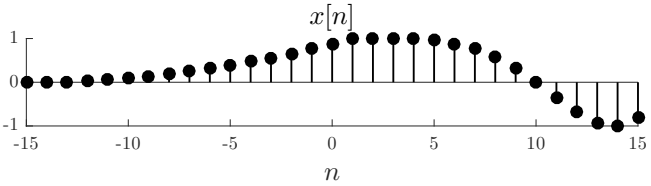
- Decimate

$$y[n] = x[2n] \quad \forall n$$

- Square time

$$y[n] = x[n^2] \quad \forall n$$

System Examples (2)



- Shift system ($m \in \mathbb{Z}$ fixed)

$$y[n] = x[n - m] \quad \forall n$$

- Moving average (combines shift, sum, scale)

$$y[n] = \frac{1}{2}(x[n] + x[n - 1]) \quad \forall n$$

- Recursive average

$$y[n] = x[n] + \alpha\, y[n - 1] \quad \forall n$$

Summary

- Systems transform one signal into another to manipulate information
- We will consider two kinds of systems:
  1. Systems that transform an infinite-length-signal x into an infinite-length signal y
  2. Systems that transform a length-N signal x into a length-N signal y

     (Such systems can also be used to process periodic signals with period N)

Linear Systems

Linear Systems

DEFINITION

A system $\mathcal{H}$ is (zero-state) **linear** if it satisfies the following two properties:

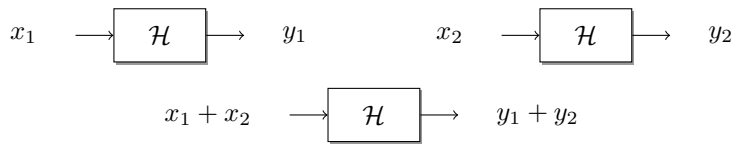
1. Scaling

$$\mathcal{H}\{\alpha\, x\} = \alpha\, \mathcal{H}\{x\} \quad \forall\ \alpha \in \mathbb{C}$$

2. Additivity

$$\text{If} \quad y_1 = \mathcal{H}\{x_1\} \quad \text{and} \quad y_2 = \mathcal{H}\{x_2\} \quad \text{then}$$

$$\mathcal{H}\{x_1 + x_2\} = y_1 + y_2$$



Linearity Notes

- A system that is not linear is called **nonlinear**
- To prove that a system is linear, you must prove rigorously that it has **both** the scaling and additivity properties for **arbitrary** input signals
- To prove that a system is nonlinear, it is sufficient to exhibit a **counterexample**

Example: Moving Average is Linear (Scaling)

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \tfrac{1}{2}(x[n] + x[n-1])$$

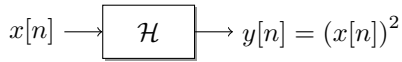
- **Scaling:** (Strategy to prove – Scale input x by $\alpha \in \mathbb{C}$, compute output y via the formula at top, and verify that it is scaled as well)
  - Let
    $$x'[n] \;=\; \alpha x[n], \quad \alpha \in \mathbb{C}$$
  - Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)
  - Then
    $$y'[n] \;=\; \frac{1}{2}(x'[n] + x'[n-1]) \;=\; \frac{1}{2}(\alpha x[n] + \alpha x[n-1]) \;=\; \alpha\left(\frac{1}{2}(x[n] + x[n-1])\right) \;=\; \alpha y[n] \;\checkmark$$

Example: Moving Average is Linear (Additivity)

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \tfrac{1}{2}(x[n] + x[n-1])$$

- **Additivity:** (Strategy to prove – Input two signals into the system and verify that the output equals the sum of the respective outputs)
  - Let
  $$x'[n] \;=\; x_1[n] + x_2[n]$$
  - Let $y'/y_1/y_2$ denote the output when $x'/x_1/x_2$ is input
  - Then
  $$\begin{aligned} y'[n] &= \frac{1}{2}(x'[n] + x'[n-1]) \;=\; \frac{1}{2}(\{x_1[n] + x_2[n]\} + \{x_1[n-1] + x_2[n-1]\}) \\ &= \frac{1}{2}(x_1[n] + x_1[n-1]) + \frac{1}{2}(x_2[n] + x_2[n-1]) \;=\; y_1[n] + y_2[n] \;\checkmark \end{aligned}$$

Example: Squaring is Nonlinear



- **Additivity:** Input two signals into the system and see what happens
  - Let
$$y_1[n] = (x_1[n])^2, \qquad y_2[n] = (x_2[n])^2$$
  - Set
$$x'[n] = x_1[n] + x_2[n]$$
  - Then
$$y'[n] = \left(x'[n]\right)^2 = (x_1[n] + x_2[n])^2 = (x_1[n])^2 + 2x_1[n]x_2[n] + (x_2[n])^2 \neq y_1[n] + y_2[n]$$
  - Nonlinear!

Linear or Nonlinear? You Be the Judge! (1)

- Identity

$$y[n] = x[n] \quad \forall n$$

- Scaling

$$y[n] = 2\,x[n] \quad \forall n$$

- Offset

$$y[n] = x[n] + 2 \quad \forall n$$

- Square signal

$$y[n] = (x[n])^2 \quad \forall n$$

- Shift

$$y[n] = x[n+2] \quad \forall n$$

- Decimate

$$y[n] = x[2n] \quad \forall n$$

- Square time

$$y[n] = x[n^2] \quad \forall n$$

Linear or Nonlinear? You Be the Judge! (2)

- Shift system ($m \in \mathbb{Z}$ fixed)

$$y[n] = x[n - m] \quad \forall n$$

- Moving average (combines shift, sum, scale)

$$y[n] = \frac{1}{2}(x[n] + x[n - 1]) \quad \forall n$$

- Recursive average

$$y[n] = x[n] + \alpha\, y[n - 1] \quad \forall n$$

Matrix Multiplication and Linear Systems

- Matrix multiplication (aka Linear Combination) is a fundamental signal processing system
- **Fact 1:** Matrix multiplications are linear systems (easy to show at home, but do it!)

$$y = \mathbf{H}\,x$$

$$y[n] = \sum_m [\mathbf{H}]_{n,m}\,x[m]$$

(Note: This formula applies for both infinite-length and finite-length signals)

- **Fact 2:** *All* linear systems can be expressed as matrix multiplications
- As a result, we will use the matrix viewpoint of linear systems extensively in the sequel
- Try at home: Express all of the linear systems in the examples above in matrix form

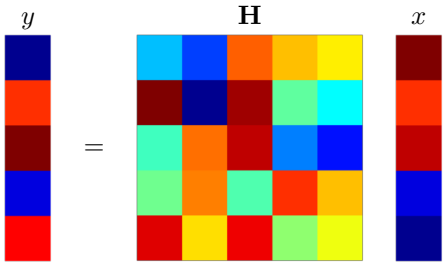
Matrix Multiplication and Linear Systems in Pictures

- Linear system

$$y \;=\; \mathbf{H}\, x$$

$$y[n] \;=\; \sum_m [\mathbf{H}]_{n,m}\, x[m] \;=\; \sum_m h_{n,m}\, x[m]$$

where $h_{n,m} \;=\; [\mathbf{H}]_{n,m}$ represents the row-n, column-m entry of the matrix $\mathbf{H}$



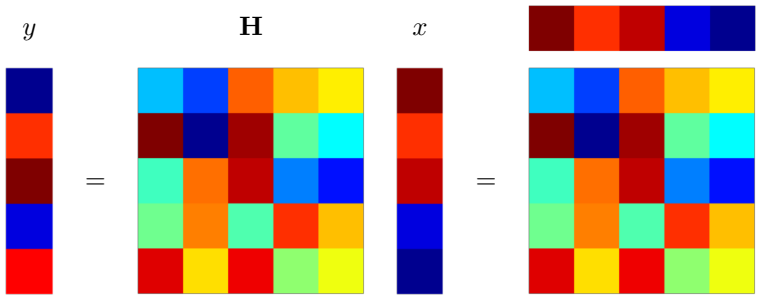
System Output as a Linear Combination of Columns

- Linear system

$$y \;=\; \mathbf{H}\, x$$

$$y[n] \;=\; \sum_m [\mathbf{H}]_{n,m}\, x[m] \;=\; \sum_m h_{n,m}\, x[m]$$

where $h_{n,m} \;=\; [\mathbf{H}]_{n,m}$ represents the row-n, column-m entry of the matrix $\mathbf{H}$



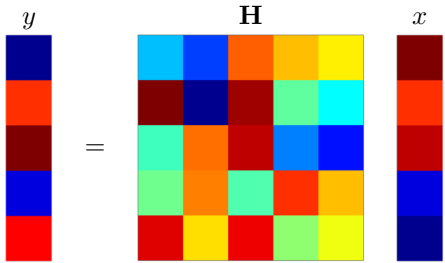
System Output as a Sequence of Inner Products

- Linear system

$$y \;=\; \mathbf{H}\, x$$

$$y[n] \;=\; \sum_m [\mathbf{H}]_{n,m}\, x[m] \;=\; \sum_m h_{n,m}\, x[m]$$

where $h_{n,m} \;=\; [\mathbf{H}]_{n,m}$ represents the row-n, column-m entry of the matrix $\mathbf{H}$



Summary

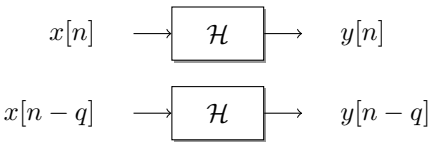
- Linear systems satisfy (1) scaling and (2) additivity
- To show a system is linear, you have to prove it rigorously assuming arbitrary inputs (work!)
- To show a system is nonlinear, you can just exhibit a counterexample (often easy!)
- Linear systems $\equiv$ matrix multiplication
  - Justifies our emphasis on linear vector spaces and matrices
  - The output signal y equals the linear combination of the columns of $\mathbf{H}$ weighted by the entries in x
  - Alternatively, the output value $y[n]$ equals the inner product between row n of $\mathbf{H}$ with x

Time-Invariant Systems

Time-Invariant Systems (Infinite-Length Signals)

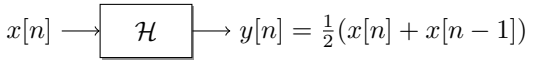
DEFINITION

A system $\mathcal{H}$ processing infinite-length signals is **time-invariant** (shift-invariant) if a time shift of the input signal creates a corresponding time shift in the output signal



- Intuition: A time-invariant system behaves the same no matter when the input is applied
- A system that is not time-invariant is called **time-varying**

Example: Moving Average is Time-Invariant



- Let

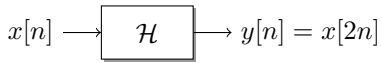
$$x'[n] = x[n-q], \quad q \in \mathbb{Z}$$

- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then

$$y'[n] \;=\; \frac{1}{2}(x'[n] + x'[n-1]) \;=\; \frac{1}{2}(x[n-q] + x[n-q-1]) \;=\; y[n-q] \;\checkmark$$

Example: Decimation is Time-Varying



- This system is time-varying; demonstrate with a counter-example
- Let
$$x'[n] = x[n-1]$$
- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)
- Then
$$y'[n] \;=\; x'[2n] \;=\; x[2n-1] \;\neq\; x[2(n-1)] \;=\; y[n-1]$$

Time-Invariant or Time-Varying? You Be the Judge! (1)

- Identity

$$y[n] = x[n] \quad \forall n$$

- Scaling

$$y[n] = 2\,x[n] \quad \forall n$$

- Offset

$$y[n] = x[n] + 2 \quad \forall n$$

- Square signal

$$y[n] = (x[n])^2 \quad \forall n$$

- Shift

$$y[n] = x[n+2] \quad \forall n$$

- Decimate

$$y[n] = x[2n] \quad \forall n$$

- Square time

$$y[n] = x[n^2] \quad \forall n$$

Time-Invariant or Time-Varying? You Be the Judge! (2)

- Shift system ($m \in \mathbb{Z}$ fixed)

$$y[n] = x[n - m] \quad \forall n$$

- Moving average (combines shift, sum, scale)

$$y[n] = \frac{1}{2}(x[n] + x[n - 1]) \quad \forall n$$

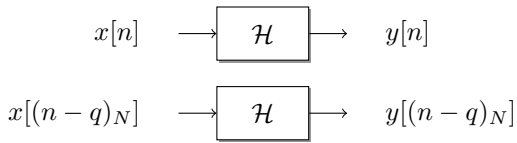
- Recursive average

$$y[n] = x[n] + \alpha\, y[n - 1] \quad \forall n$$

Time-Invariant Systems (Finite-Length Signals)

DEFINITION

A system $\mathcal{H}$ processing length-N signals is **time-invariant** (shift-invariant) if a circular time shift of the input signal creates a corresponding circular time shift in the output signal



- Intuition: A time-invariant system behaves the same no matter when the input is applied
- A system that is not time-invariant is called **time-varying**

Summary

- Time-invariant systems behave the same no matter when the input is applied
- Infinite-length signals: Invariance with respect to any integer time shift
- Finite-length signals: Invariance with respect to a circular time shift
- To show a system is time-invariant, you have to prove it rigorously assuming arbitrary inputs (work!)
- To show a system is time-varying, you can just exhibit a counterexample (often easy!)

Linear Time-Invariant Systems

Linear Time Invariant (LTI) Systems

DEFINITION

A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI systems are the foundation of signal processing and the main subject of this course

LTI or Not? You Be the Judge! (1)

- Identity

$$y[n] = x[n] \quad \forall n$$

- Scaling

$$y[n] = 2\,x[n] \quad \forall n$$

- Offset

$$y[n] = x[n] + 2 \quad \forall n$$

- Square signal

$$y[n] = (x[n])^2 \quad \forall n$$

- Shift

$$y[n] = x[n+2] \quad \forall n$$

- Decimate

$$y[n] = x[2n] \quad \forall n$$

- Square time

$$y[n] = x[n^2] \quad \forall n$$

LTI or Not? You Be the Judge! (2)

- Shift system ($m \in \mathbb{Z}$ fixed)

$$y[n] = x[n - m] \quad \forall n$$

- Moving average (combines shift, sum, scale)

$$y[n] = \frac{1}{2}(x[n] + x[n - 1]) \quad \forall n$$

- Recursive average

$$y[n] = x[n] + \alpha\, y[n - 1] \quad \forall n$$

Matrix Multiplication and LTI Systems (Infinite-Length Signals)

- Recall that all linear systems can be expressed as matrix multiplications

$$y = \mathbf{H}\, x$$

$$y[n] = \sum_m [\mathbf{H}]_{n,m}\, x[m]$$

  Here $\mathbf{H}$ is a matrix with infinitely many rows and columns

- Let $h_{n,m} = [\mathbf{H}]_{n,m}$ represent the row-n, column-m entry of the matrix $\mathbf{H}$

$$y[n] = \sum_m h_{n,m}\, x[m]$$

- When the linear system is also shift invariant, $\mathbf{H}$ has a special structure

Matrix Structure of LTI Systems (Infinite-Length Signals)

- Linear system for infinite-length signals can be expressed as

$$y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; \sum_{m=-\infty}^{\infty} h_{n,m}\, x[m], \quad -\infty < n < \infty$$

- Enforcing time invariance implies that for all $q \in \mathbb{Z}$

$$\mathcal{H}\{x[n-q]\} \;=\; \sum_{m=-\infty}^{\infty} h_{n,m}\, x[m-q] \;=\; y[n-q]$$

- Change of variables: $n' = n - q$ and $m' = m - q$

$$\mathcal{H}\{x[n']\} \;=\; \sum_{m'=-\infty}^{\infty} h_{n'+q,m'+q}\, x[m'] \;=\; y[n']$$

- Comparing first and third equations, we see that for an LTI system

$$h_{n,m} \;=\; h_{n+q,m+q} \quad \forall\, q \in \mathbb{Z}$$

LTI Systems are Toeplitz Matrices (Infinite-Length Signals) (1)

- For an LTI system with infinite-length signals

$$h_{n,m} \;=\; h_{n+q,m+q} \quad \forall\, q \in \mathbb{Z}$$

$$\mathbf{H} \;=\; \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & h_{-1,-1} & h_{-1,0} & h_{-1,1} & \cdots \\ \cdots & h_{0,-1} & h_{0,0} & h_{0,1} & \cdots \\ \cdots & h_{1,-1} & h_{1,0} & h_{1,1} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} \;=\; \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & h_{0,0} & h_{-1,0} & h_{-2,0} & \cdots \\ \cdots & h_{1,0} & h_{0,0} & h_{-1,0} & \cdots \\ \cdots & h_{2,0} & h_{1,0} & h_{0,0} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

- Entries on the matrix diagonals are the same – **Toeplitz matrix**

LTI Systems are Toeplitz Matrices (Infinite-Length Signals) (2)

- All of the entries in a Toeplitz matrix can be expressed in terms of the entries of the
  - 0-**th column**: $h[n] = h_{n,0}$
  - **Time-reversed** 0-**th row**: $h[m] = h_{0,-m}$

$$\mathbf{H} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & h_{0,0} & h_{-1,0} & h_{-2,0} & \cdots \\ \cdots & h_{1,0} & h_{0,0} & h_{-1,0} & \cdots \\ \cdots & h_{2,0} & h_{1,0} & h_{0,0} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & h[0] & h[-1] & h[-2] & \cdots \\ \cdots & h[1] & h[0] & h[-1] & \cdots \\ \cdots & h[2] & h[1] & h[0] & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

- Row-n, column-m entry of the matrix $[\mathbf{H}]_{n,m} = h_{n,m} = h[n-m]$

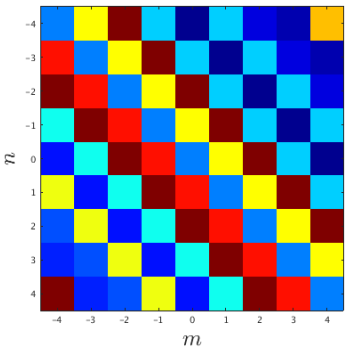
LTI Systems are Toeplitz Matrices (Infinite-Length Signals) (3)

- All of the entries in a Toeplitz matrix can be expressed in terms of the entries of the
  - 0-**th column**: $h[n] = h_{n,0}$ (this is an infinite-length signal/column vector; call it h)
  - **Time-reversed** 0-**th row**: $h[m] = h_{0,-m}$

- Example: Snippet of a Toeplitz matrix

$$[\mathbf{H}]_{n,m} = h_{n,m} = h[n-m]$$

$$\mathbf{H} =$$



$$h =$$

- Note the diagonals!

Matrix Structure of LTI Systems (Finite-Length Signals)

- Linear system for signals of length N can be expressed as

$$y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; \sum_{m=0}^{N-1} h_{n,m}\, x[m], \quad 0 \le n \le N-1$$

- Enforcing time invariance implies that for all $q \in \mathbb{Z}$

$$\mathcal{H}\{x[(n-q)_N]\} \;=\; \sum_{m=0}^{N-1} h_{n,m}\, x[(m-q)_N] \;=\; y[(n-q)_N]$$

- Change of variables: $n' = n - q$ and $m' = m - q$

$$\mathcal{H}\{x[(n')_N]\} \;=\; \sum_{m'=-q}^{N-1-q} h_{(n'+q)_N,(m'+q)_N}\, x[(m')_N] \;=\; y[(n')_N]$$

- Comparing first and third equations, we see that for an LTI system

$$h_{n,m} \;=\; h_{(n+q)_N,(m+q)_N} \quad \forall\, q \in \mathbb{Z}$$

LTI Systems are Circulant Matrices (Finite-Length Signals) (1)

- For an LTI system with length-N signals

$$h_{n,m} \;=\; h_{(n+q)_N,(m+q)_N} \quad \forall\, q \in \mathbb{Z}$$

$$\mathbf{H} \;=\; \begin{bmatrix} h_{0,0} & h_{0,1} & h_{0,2} & \cdots & h_{0,N-1} \\ h_{1,0} & h_{1,1} & h_{1,2} & \cdots & h_{1,N-1} \\ h_{2,0} & h_{2,1} & h_{2,2} & \cdots & h_{2,N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ h_{N-1,0} & h_{N-1,1} & h_{N-1,2} & \cdots & h_{N-1,N-1} \end{bmatrix} \;=\; \begin{bmatrix} h_{0,0} & h_{N-1,0} & h_{N-2,0} & \cdots & h_{1,0} \\ h_{1,0} & h_{0,0} & h_{N-1,0} & \cdots & h_{2,0} \\ h_{2,0} & h_{1,0} & h_{0,0} & \cdots & h_{3,0} \\ \vdots & \vdots & \vdots & & \vdots \\ h_{N-1,0} & h_{N-2,0} & h_{N-3,0} & \cdots & h_{0,0} \end{bmatrix}$$

- Entries on the matrix diagonals are the same + circular wraparound – **circulant matrix**

LTI Systems are Circulant Matrices (Finite-Length Signals) (2)

- All of the entries in a circulant matrix can be expressed in terms of the entries of the
  - 0-**th column**: $h[n] = h_{n,0}$
  - **Circularly time-reversed** 0-**th row**: $h[m] = h_{0,(-m)_N}$

$$\mathbf{H} = \begin{bmatrix} h_{0,0} & h_{N-1,0} & h_{N-2,0} & \cdots & h_{1,0} \\ h_{1,0} & h_{0,0} & h_{N-1,0} & \cdots & h_{2,0} \\ h_{2,0} & h_{1,0} & h_{0,0} & \cdots & h_{3,0} \\ \vdots & \vdots & \vdots & & \vdots \\ h_{N-1,0} & h_{N-2,0} & h_{N-3,0} & \cdots & h_{0,0} \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix}$$

- Row-n, column-m entry of the matrix $[\mathbf{H}]_{n,m} = h_{n,m} = h[(n-m)_N]$

LTI Systems are Circulant Matrices (Finite-Length Signals) (3)

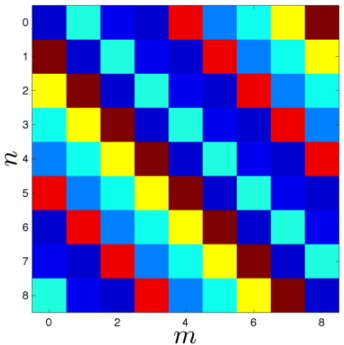
- All of the entries in a circulant matrix can be expressed in terms of the entries of the
  - 0-**th column**: $h[n] = h_{n,0}$ (this is a signal/column vector; call it h)
  - **Circularly time-reversed** 0-**th row**: $h[m] = h_{0,-m}$

- Example: circulant matrix

$$[\mathbf{H}]_{n,m} = h_{n,m} = h[(n-m)_N]$$

- Note the diagonals and circulant shifts!

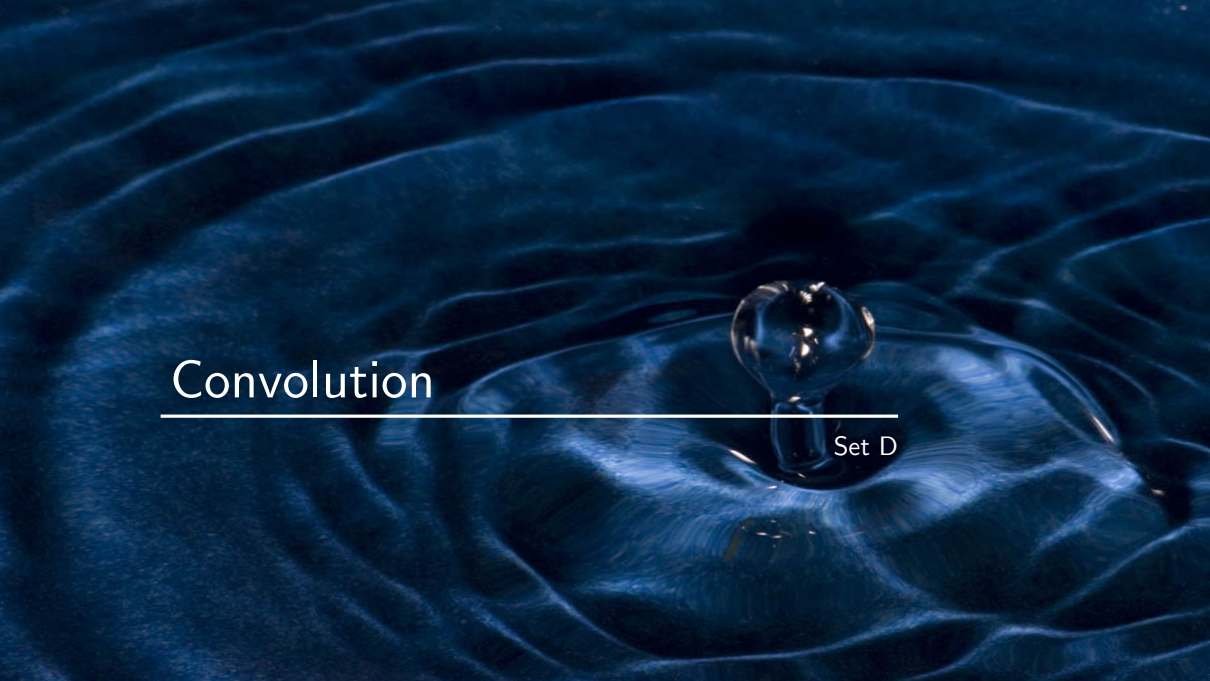
$\mathbf{H} =$



$h =$

Summary

- LTI = Linear + Time-Invariant
- Fundamental signal processing system (and our focus for the rest of the course)
- Infinite-length signals: System = Toeplitz matrix $\mathbf{H}$
  - $[\mathbf{H}]_{n,m} = h_{n,m} = h[n-m]$
- Finite-length signals: System = circulant matrix $\mathbf{H}$
  - $[\mathbf{H}]_{n,m} = h_{n,m} = h[(n-m)_N]$

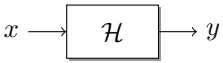


Convolution

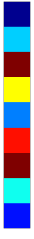
Set D

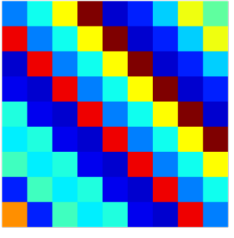
Impulse Response

Recall: LTI Systems are Toeplitz Matrices (Infinite-Length Signals)



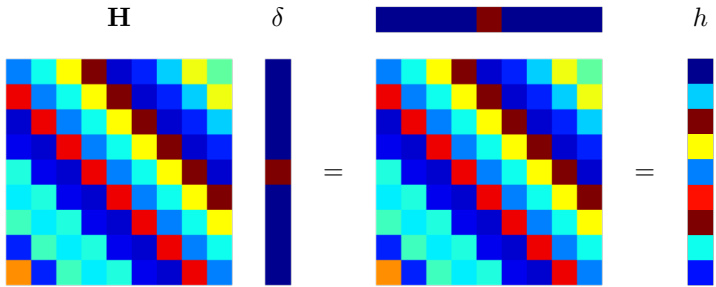
- LTI system $=$ multiplication by infinitely large Toeplitz matrix $\mathbf{H}$: $y = \mathbf{H}x$
- All of the entries in $\mathbf{H}$ can be obtained from the
  - 0-**th column**: $h[n] = h_{n,0}$ (this is a signal/column vector; call it h)
  - **Time-reversed** 0-**th row**: $h[m] = h_{0,-m}$
- $[\mathbf{H}]_{n,m} = h_{n,m}$
  $= h[n-m]$
- Columns/rows of $\mathbf{H}$ are shifted versions of the 0-th column/row

$h =$ 

$\mathbf{H} =$ 

Impulse Response (Infinite-Length Signals)

- The 0-th column of the matrix $\mathbf{H}$ – the column vector h – has a special interpretation
- Compute the output when the input is a **delta function** (impulse): $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$



- This suggests that we call h the **impulse response** of the system

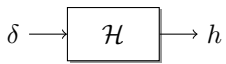
Impulse Response from Formulas (Infinite-Length Signals)

- General formula for LTI matrix multiplication

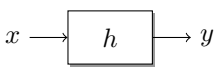
$$y[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- Let the input $x[n] = \delta[n]$ and compute

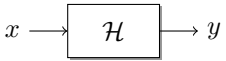
$$\sum_{m=-\infty}^{\infty} h[n-m]\, \delta[m] \;=\; h[n] \;\checkmark$$



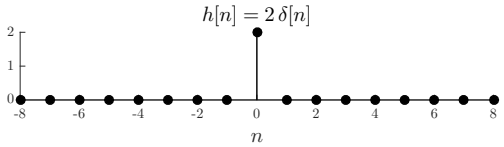
- The impulse response characterizes an LTI system
  (that is, carries all of the information contained in the matrix $\mathbf{H}$)



Example: Impulse Response of the Scaling System



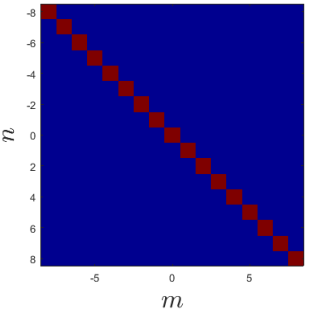
- Consider system for infinite-length signals; finite-length signal case is similar



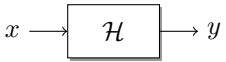
- Scaling system: $y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; 2\,x[n]$
- Impulse response: $h[n] \;=\; \mathcal{H}\{\delta[n]\} \;=\; 2\,\delta[n]$
- Toeplitz system matrix:

$$[\mathbf{H}]_{n,m} = h[n-m] = 2\,\delta[n-m]$$

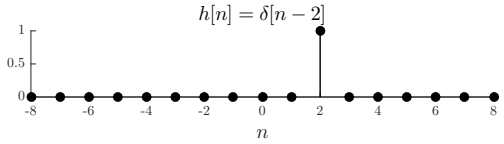
$\mathbf{H} =$



Example: Impulse Response of the Shift System



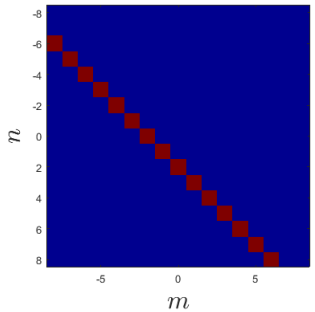
- Consider system for infinite-length signals; finite-length signal case uses circular shift



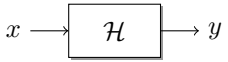
- Shift system: $y[n] = \mathcal{H}\{x[n]\} = x[n-2]$
- Impulse response: $h[n] = \mathcal{H}\{\delta[n]\} = \delta[n-2]$
- Toeplitz system matrix:

$$[\mathbf{H}]_{n,m} = h[n-m] = \delta[n-m-2]$$

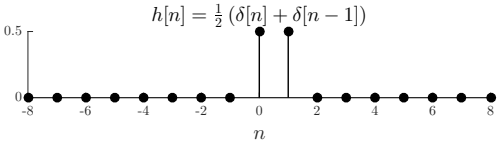
$\mathbf{H} =$



Example: Impulse Response of the Moving Average System



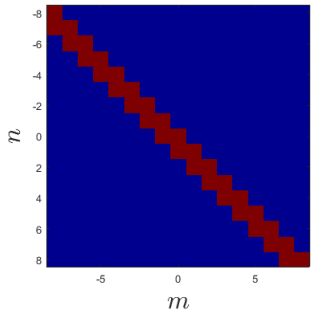
- Consider system for infinite-length signals; finite-length signal case is similar



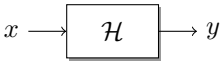
- Moving average system: $y[n] = \mathcal{H}\{x[n]\} = \frac{1}{2}(x[n] + x[n-1])$
- Impulse response: $h[n] = \mathcal{H}\{\delta[n]\} = \frac{1}{2}(\delta[n] + \delta[n-1])$
- Toeplitz system matrix:

$$[\mathbf{H}]_{n,m} = h[n-m] = \frac{1}{2}(\delta[n-m] + \delta[n-m-1])$$

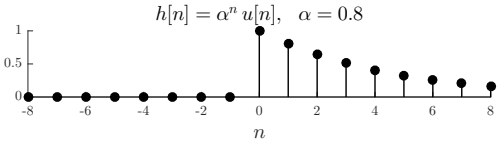
$\mathbf{H} =$



Example: Impulse Response of the Recursive Average System



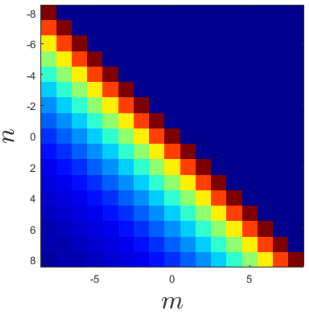
- Consider system for infinite-length signals; finite-length signal case is similar



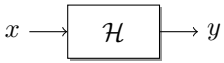
- Recursive average system: $y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; x[n] + \alpha\, y[n-1]$
- Impulse response: $h[n] \;=\; \mathcal{H}\{\delta[n]\} \;=\; \alpha^n\, u[n]$
- Toeplitz system matrix:

$$[\mathbf{H}]_{n,m} = h[n-m] = \alpha^{n-m}\, u[n-m]$$

$\mathbf{H} \;=$



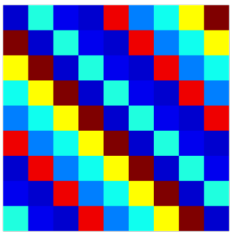
Recall: LTI Systems are Circulant Matrices (Finite-Length Signals)



- LTI system $=$ multiplication by $N \times N$ circulant matrix $\mathbf{H}$: $y = \mathbf{H}x$
- All of the entries in $\mathbf{H}$ can be obtained from the
  - 0-**th column**: $h[n] = h_{n,0}$ (this is a signal/column vector; call it h)
  - **Time-reversed** 0-**th row**: $h[m] = h_{0,(-m)_N}$
- $[\mathbf{H}]_{n,m} = h_{n,m}$
  $= h[(n-m)_N]$
- Columns/rows of $\mathbf{H}$ are circularly shifted versions of the 0-th column/row

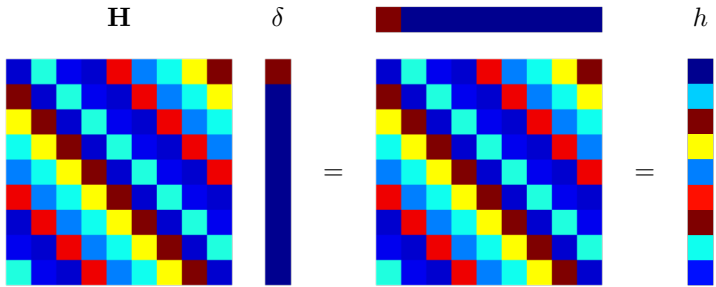
$h =$

$\mathbf{H} =$



Impulse Response (Finite-Length Signals)

- The 0-th column of the matrix $\mathbf{H}$ – the column vector h – has a special interpretation
- Compute the output when the input is a **delta function** (impulse): $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$



- This suggests that we call h the **impulse response** of the system

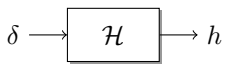
Impulse Response from Formulas (Finite-Length Signals)

- General formula for LTI matrix multiplication

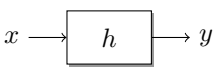
$$y[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m]$$

- Let the input $x[n] = \delta[n]$ and compute

$$\sum_{m=0}^{N-1} h[(n-m)_N]\, \delta[m] \;=\; h[n] \;\checkmark$$

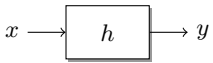


- The impulse response **characterizes** an LTI system
  (that is, carries all of the information contained in the matrix $\mathbf{H}$)



Summary

- LTI system $=$ multiplication by infinite-sized Toeplitz or $N \times N$ circulant matrix $\mathbf{H}$: $y = \mathbf{H}x$
- The **impulse response** h of an LTI system $=$ the response to an impulse δ
  - The impulse response is the 0-th column of the matrix $\mathbf{H}$
  - The impulse response characterizes an LTI system

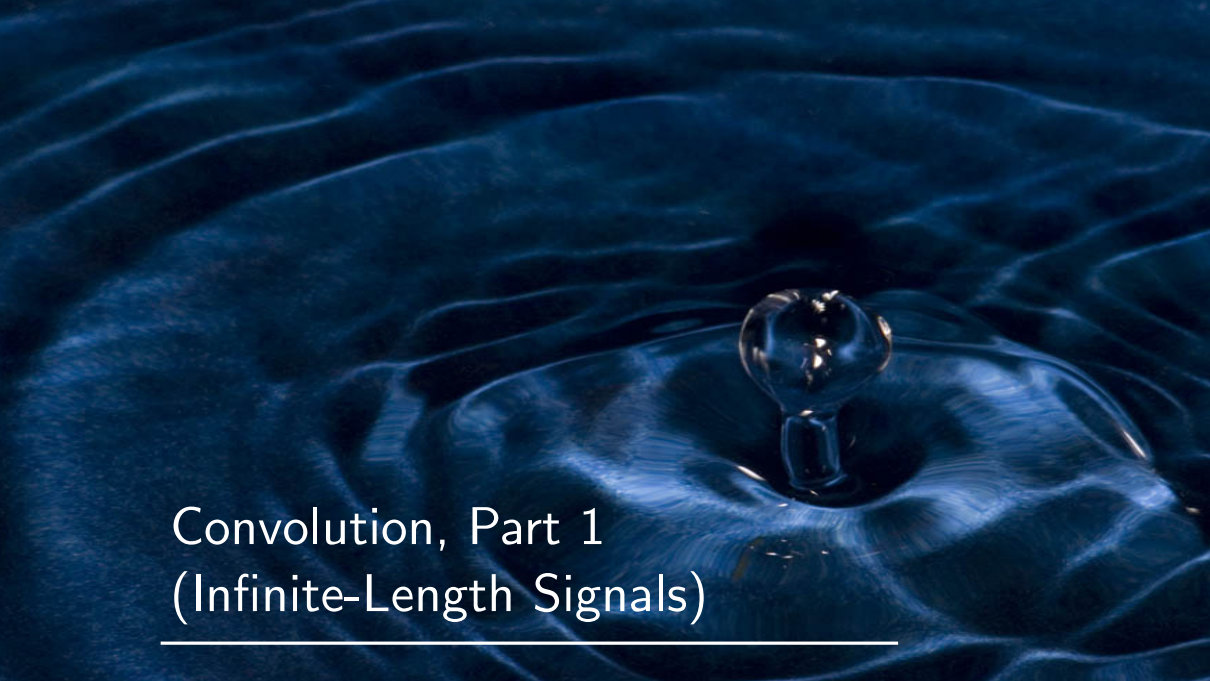


- Formula for the output signal y in terms of the input signal x and the impulse response h
  - Infinite-length signals

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m], \quad -\infty < n < \infty$$

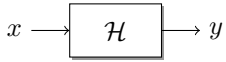
  - Length-N signals

$$y[n] = \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m], \quad 0 \le n \le N-1$$



Convolution, Part 1 (Infinite-Length Signals)

Three Ways to Compute the Output of an LTI System Given the Input



1. If $\mathcal{H}$ is defined in terms of a formula or **algorithm**, apply the input x and compute $y[n]$ at each time point $n \in \mathbb{Z}$
   - This is how systems are usually applied in computer code and hardware

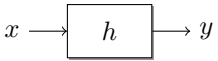
2. Find the impulse response h (by inputting $x[n] = \delta[n]$), form the **Toeplitz system matrix** $\mathbf{H}$, and multiply by the (infinite-length) input signal vector x to obtain $y = \mathbf{H}\,x$
   - This is not usually practical but is useful for conceptual purposes

3. Find the impulse response h and apply the formula for matrix/vector product for each $n \in \mathbb{Z}$

$$y[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\,x[m] \;=\; x[n] * h[n]$$

   - This is called **convolution** and is both conceptually and practically useful (Matlab command: `conv`)

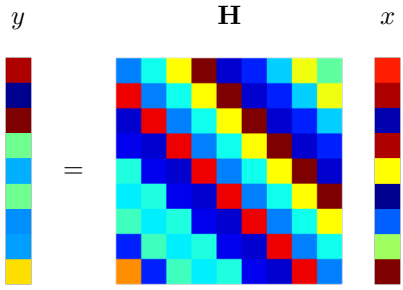
Convolution as a Sequence of Inner Products (1)



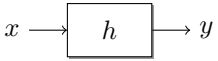
- Convolution formula

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- To compute the entry $y[n]$ in the output vector y:
  1. **Time reverse** the impulse response vector h and **shift** it n time steps to the right (delay)
  2. Compute the **inner product** between the shifted impulse response and the input vector x
- Repeat for every n



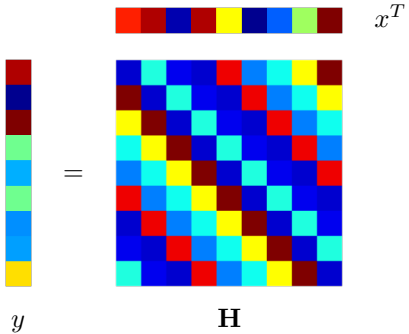
Convolution as a Sequence of Inner Products (2)



- Convolution formula

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- To compute the entry $y[n]$ in the output vector y:
  1. **Time reverse** the impulse response vector h and **shift** it n time steps to the right (delay)
  2. Compute the **inner product** between the shifted impulse response and the input vector x
- Repeat for every n



Recall: Row n of $\mathbf{H}$ is time-reversed impulse response shifted right by n

A Seven-Step Program for Computing Convolution By Hand

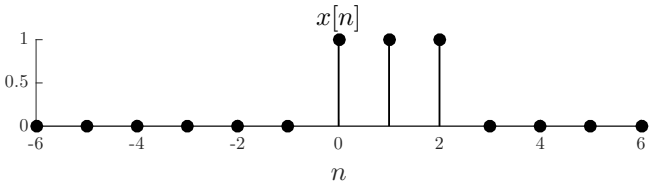
$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- **Step 1:** Decide which of x or h you will flip and shift; you have a choice since $x * h = h * x$
- **Step 2:** Plot $x[m]$ as a function of m
- **Step 3:** Plot the time-reversed impulse response $h[-m]$
- **Step 4:** To compute y at the time point n, plot the time-reversed impulse response after it has been shifted to the right (delayed) by n time units: $h[-(m-n)] = h[n-m]$
- **Step 5:** $y[n] =$ the inner product between the signals $x[m]$ and $h[n-m]$
  (Note: for complex signals, do not complex conjugate the second signal in the inner product)
- **Step 6:** Repeat for all n of interest (potentially all $n \in \mathbb{Z}$)
- **Step 7:** Plot $y[n]$ and perform a reality check to make sure your answer seems reasonable

First Convolution Example (1)

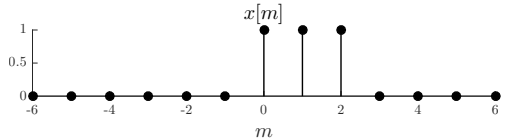
$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

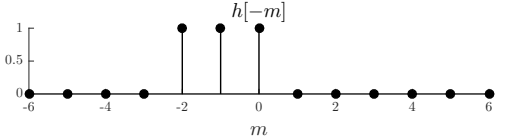
- Convolve a unit pulse with itself



First Convolution Example (2)

$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$





Convolution, Part 2 (Infinite-Length Signals)

A Seven-Step Program for Computing Convolution By Hand

$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

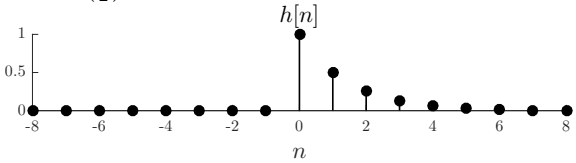
- **Step 1:** Decide which of x or h you will flip and shift; you have a choice since $x * h = h * x$
- **Step 2:** Plot $x[m]$ as a function of m
- **Step 3:** Plot the time-reversed impulse response $h[-m]$
- **Step 4:** To compute y at the time point n, plot the time-reversed impulse response after it has been shifted to the right (delayed) by n time units: $h[-(m-n)] = h[n-m]$
- **Step 5:** $y[n] =$ the inner product between the signals $x[m]$ and $h[n-m]$
  (Note: for complex signals, do not complex conjugate the second signal in the inner product)
- **Step 6:** Repeat for all n of interest (potentially all $n \in \mathbb{Z}$)
- **Step 7:** Plot $y[n]$ and perform a reality check to make sure your answer seems reasonable

Second Convolution Example (1)

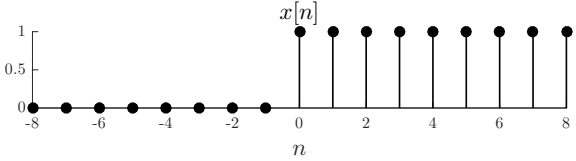
- Recall the **recursive average system**

$$y[n] = x[n] + \frac{1}{2}\, y[n-1]$$

and its impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$

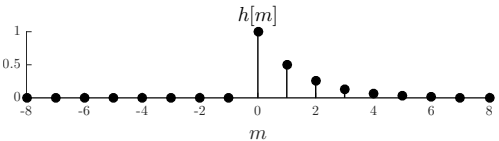


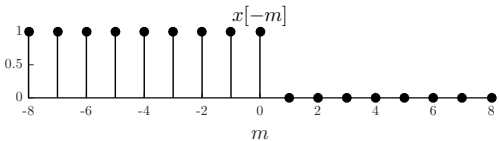
- Compute the output y when the input is a unit step $x[n] = u[n]$



Second Convolution Example (2)

$$y[n] \;=\; h[n] * x[n] \;=\; \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$



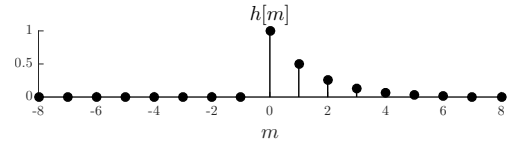


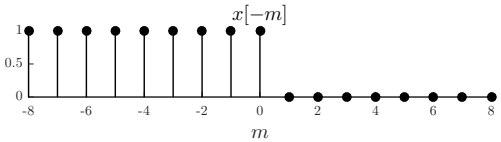
- Recall the super useful formula for the **finite geometric series**

$$\sum_{k=N_1}^{N_2} a^k \;=\; \frac{a^{N_1} - a^{N_2+1}}{1-a}, \quad N_1 \le N_2$$

Second Convolution Example (3)

$$y[n] \;=\; h[n] * x[n] \;=\; \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$



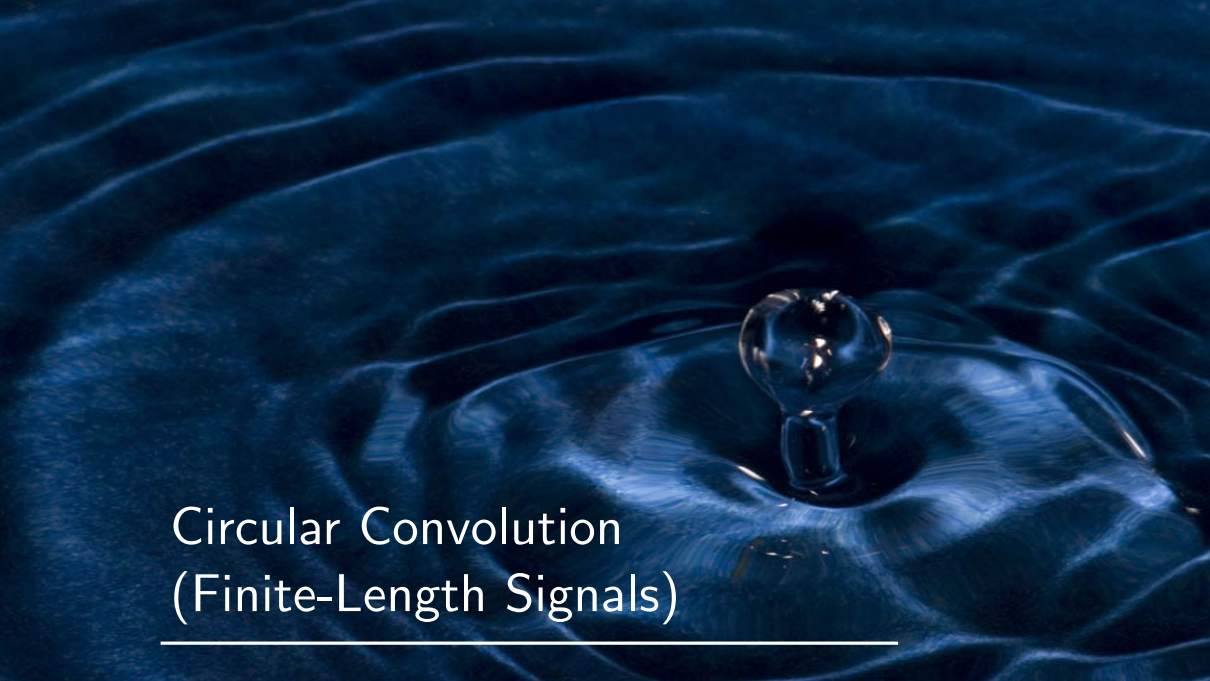


Summary

- **Convolution** formula for the output y of an LTI system given the input x and the impulse response h (infinite-length signals)

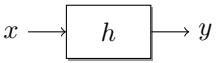
$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- Convolution is a sequence of inner products between the signal and the shifted, time-reversed impulse response
- Seven-step program for computing convolution by hand
- Check your work and compute large convolutions using Matlab command `conv`
- Practice makes perfect!



Circular Convolution (Finite-Length Signals)

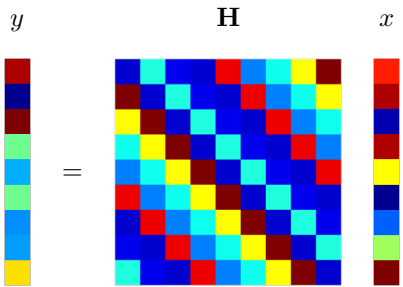
Circular Convolution as a Sequence of Inner Products (1)



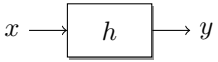
- Convolution formula

$$y[n] \;=\; x[n] \circledast h[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m]$$

- To compute the entry $y[n]$ in the output vector y:
  1. **Circularly time reverse** the impulse response vector h and **circularly shift** it n time steps to the right (delay)
  2. Compute the **inner product** between the shifted impulse response and the input vector x
- Repeat for every n



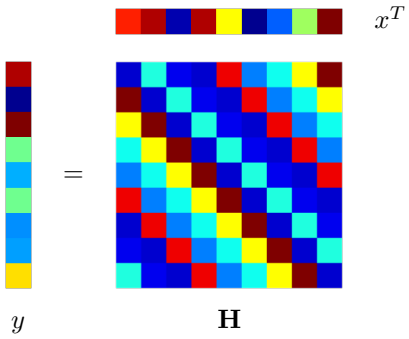
Circular Convolution as a Sequence of Inner Products (2)



- Convolution formula

$$y[n] \;=\; x[n] \circledast h[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m]$$

- To compute the entry $y[n]$ in the output vector y:
  1. **Circularly time reverse** the impulse response vector h and **circularly shift** it n time steps to the right (delay)
  2. Compute the **inner product** between the shifted impulse response and the input vector x
- Repeat for every n



Recall: Row n of $\mathbf{H}$ is circularly time-reversed impulse response circularly shifted right by n

A Seven-Step Program for Computing Circular Convolution By Hand

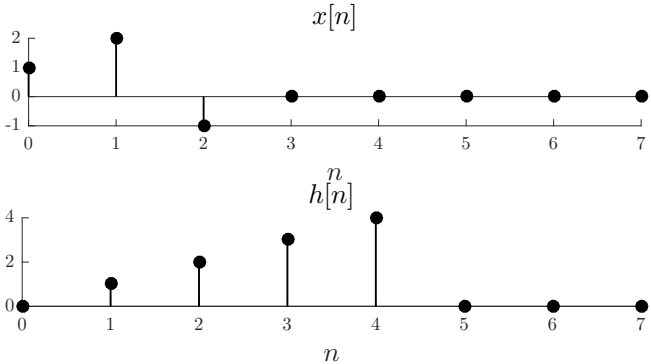
$$y[n] \;=\; x[n] \circledast h[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m]$$

- **Step 1:** Decide which of x or h you will flip and shift; you have a choice since $x * h = h * x$
- **Step 2:** Plot $x[m]$ as a function of m on a wheel with N time locations
- **Step 3:** Plot the circularly time-reversed impulse response $h[(-m)_N]$ on a clock with N time locations
- **Step 4:** To compute y at the time point n, plot the time-reversed impulse response after it has been shifted counter-clockwise (delayed) by n time units: $h[(-(m-n))_N] = h[(n-m)_N]$
- **Step 5:** $y[n] =$ the inner product between the signals $x[m]$ and $h[(n-m)_N]$
  (Note: for complex signals, do not complex conjugate the second signal in the inner product)
- **Step 6:** Repeat for all $n = 0, 1, \ldots, N-1$
- **Step 7:** Plot $y[n]$ and perform a reality check to make sure your answer seems reasonable

Circular Convolution Example

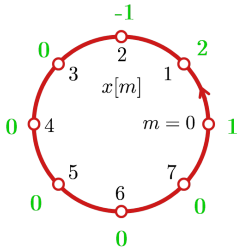
$$y[n] \;=\; x[n] \circledast h[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m]$$

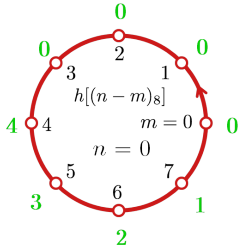
- For $N = 8$, circularly convolve x and h

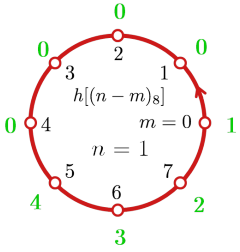


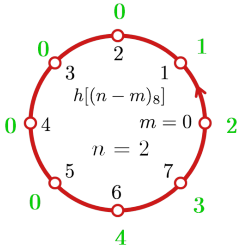
Circular Convolution Example

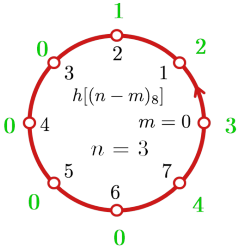
$$y[n] \;=\; x[n] \circledast h[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m]$$





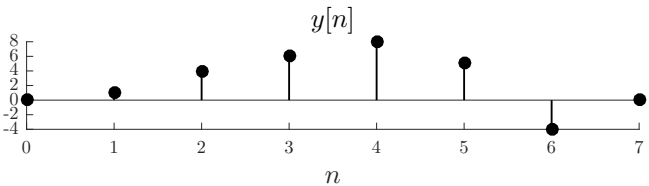






Circular Convolution Example

$$y[n] \;=\; x[n] \circledast h[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m]$$

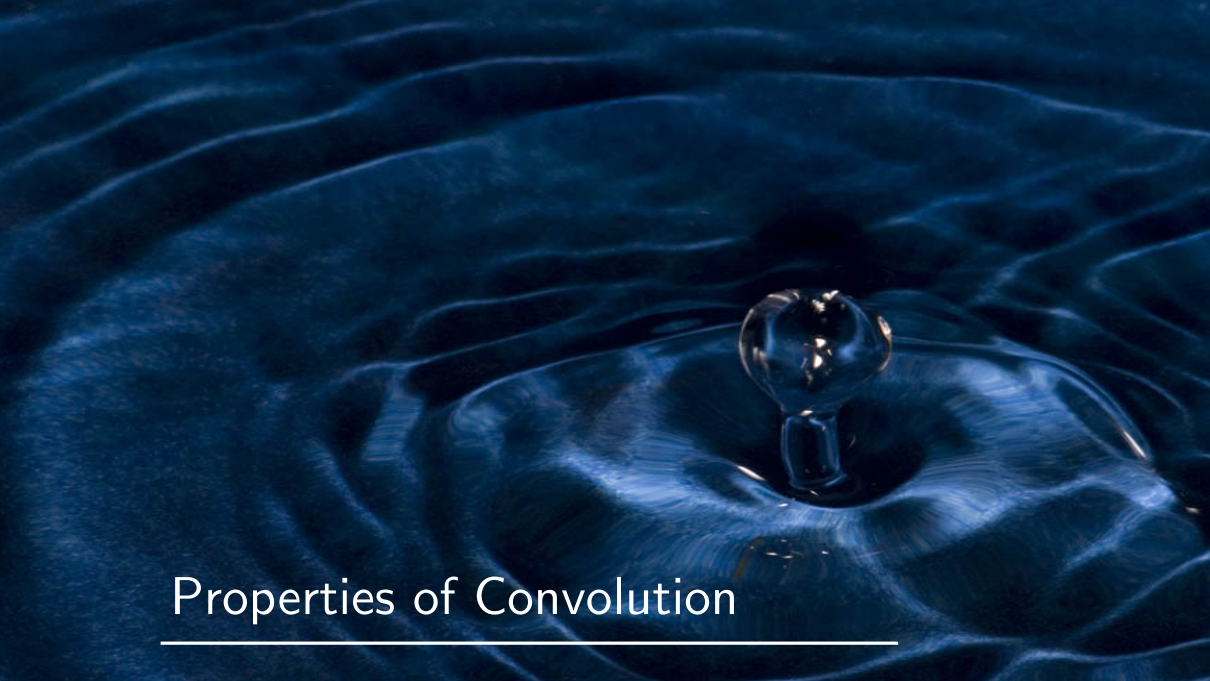


Summary

- **Circular convolution** formula for the output y of an LTI system given the input x and the impulse response h (length-N signals)

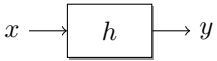
$$y[n] \;=\; x[n] \circledast h[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m]$$

- Circular convolution is a sequence of inner products between the signal and the circularly shifted, time-reversed impulse response
- Seven-step program for computing circular convolution by hand
- Check your work and compute large circular convolutions using Matlab command `cconv`
- Practice makes perfect!



Properties of Convolution

Properties of Convolution



- Input signal x, LTI system impulse response h, and output signal y are related by the **convolution**
  - Infinite-length signals

$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m], \quad -\infty < n < \infty$$

  - Length-N signals

$$y[n] \;=\; x[n] \circledast h[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m], \quad 0 \le n \le N-1$$

- Thanks to the Toeplitz/circulant structure of LTI systems, convolution has very special properties
- We will emphasize infinite-length convolution, but similar arguments hold for circular convolution except where noted

Convolution is Commutative

- **Fact:** Convolution is commutative: $x * h \;=\; h * x$
- These block diagrams are equivalent:

- Enables us to pick either h or x to flip and shift (or stack into a matrix) when convolving
- To prove, start with the convolution formula

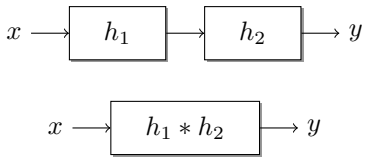
$$y[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\,x[m] \;=\; x[n] * h[n]$$

and change variables to $k = n - m \;\Rightarrow\; m = n - k$

$$y[n] \;=\; \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] \;=\; h[n] * x[n] \;\checkmark$$

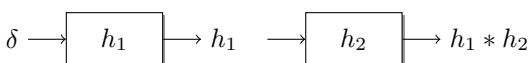
Cascade Connection of LTI Systems

- Impulse response of the **cascade** (aka series connection) of two LTI systems: $y = \mathbf{H}_1\mathbf{H}_2\, x$



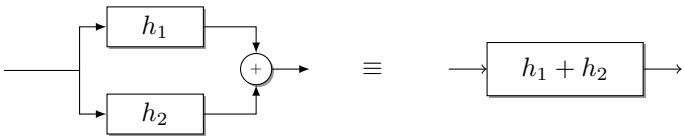
- Interpretation: The product of two Toeplitz/circulant matrices is a Toeplitz/circulant matrix

- Easy proof by picture; find impulse response the old school way



Parallel Connection of LTI Systems

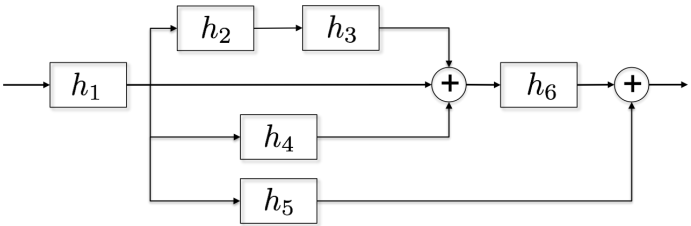
- Impulse response of the **parallel connection** of two LTI systems $y = (\mathbf{H}_1 + \mathbf{H}_2)\, x$

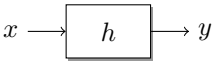


- Proof is an easy application of the linearity of an LTI system

Example: Impulse Response of a Complicated Connection of LTI Systems

- Compute the overall effective impulse response of the following system





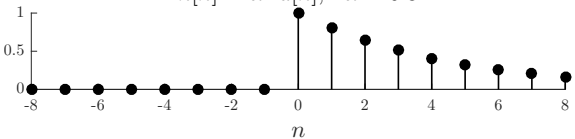
Causal Systems

DEFINITION

A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- **Fact:** An LTI system is causal if its impulse response is causal: $h[n] = 0$ for $n < 0$

$$h[n] = \alpha^n \, u[n], \quad \alpha = 0.8$$



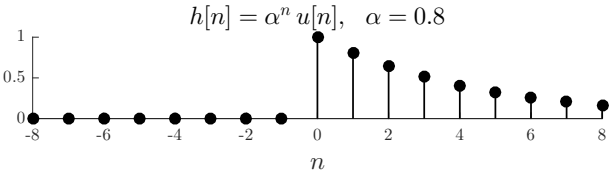
- To prove, note that the convolution

$$y[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

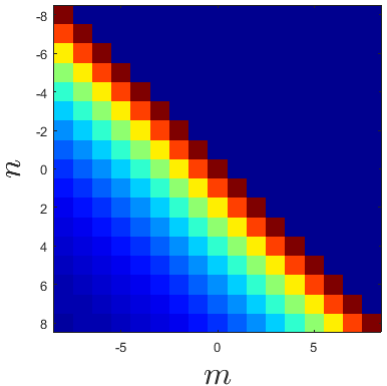
does not look into the future if $h[n-m] = 0$ when $m > n$; equivalently, $h[n'] = 0$ when $n' < 0$

Causal System Matrix

- **Fact:** An LTI system is causal if its impulse response is causal: $h[n] = 0$ for $n < 0$



- Toeplitz system matrix is **lower triangular**

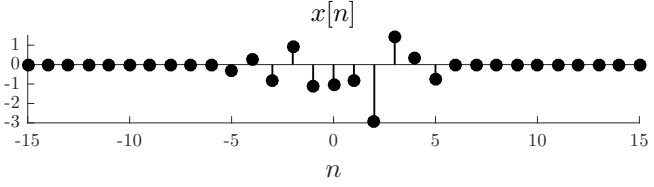


Duration of Convolution

DEFINITION

The signal x has **support interval** $[N_1, N_2]$, $N_1 \leq N_2$, if $x[n] = 0$ for all $n < N_1$ and $n > N_2$. The **duration** D_x of x equals $N_2 - N_1 + 1$

- Example: A signal with support interval $[-5, 5]$ and duration 11 samples



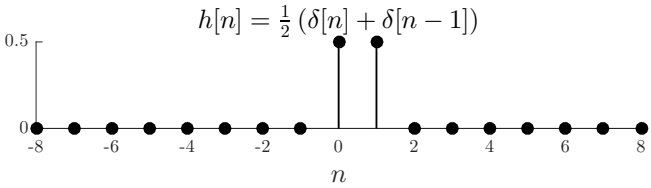
- **Fact:** If x has duration D_x samples and h has duration D_h samples, then the convolution $y = x * h$ has duration at most $D_x + D_h - 1$ samples (proof by picture is simple)

Duration of Impulse Response – FIR

DEFINITION

An LTI system has a **finite impulse response** (FIR) if the duration of its impulse response h is finite

- Example: Moving average system $\quad y[n] = \mathcal{H}\{x[n]\} = \frac{1}{2}\left(x[n] + x[n-1]\right)$

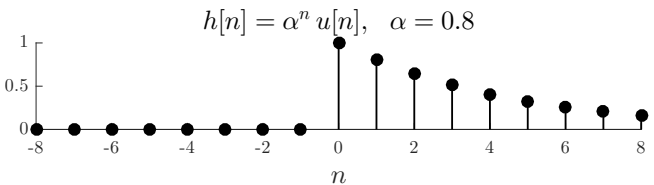


Duration of Impulse Response – IIR

DEFINITION

An LTI system has an **infinite impulse response** (IIR) if the duration of its impulse response h is infinite

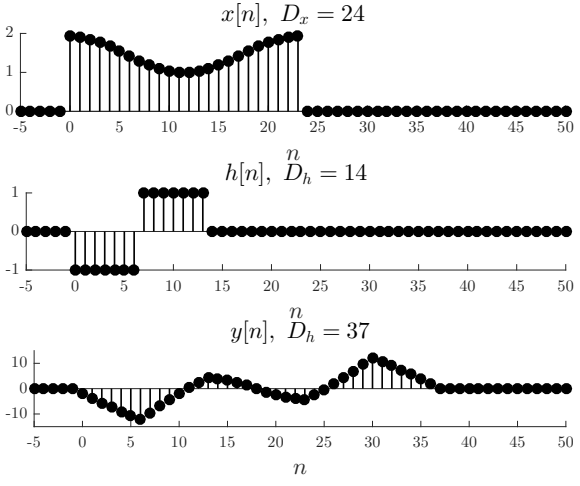
- Example: Recursive average system $\quad y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; x[n] + \alpha\, y[n-1]$



- Note: Obviously the FIR/IIR distinction applies only to infinite-length signals

Duration of Convolution

- Recall that, if x has duration D_x samples and h has duration D_h samples, then the infinite-length convolution $y = x * h$ has duration at most $D_x + D_h - 1$ samples (proof by picture is simple)



Implementing Infinite-Length Convolution with Circular Convolution

- Consider two infinite-length signals: x has duration D_x samples and h has duration D_h samples, $D_x, D_h < \infty$

- Recall that their infinite-length convolution $y = x * h$ has duration at most $D_x + D_h - 1$ samples

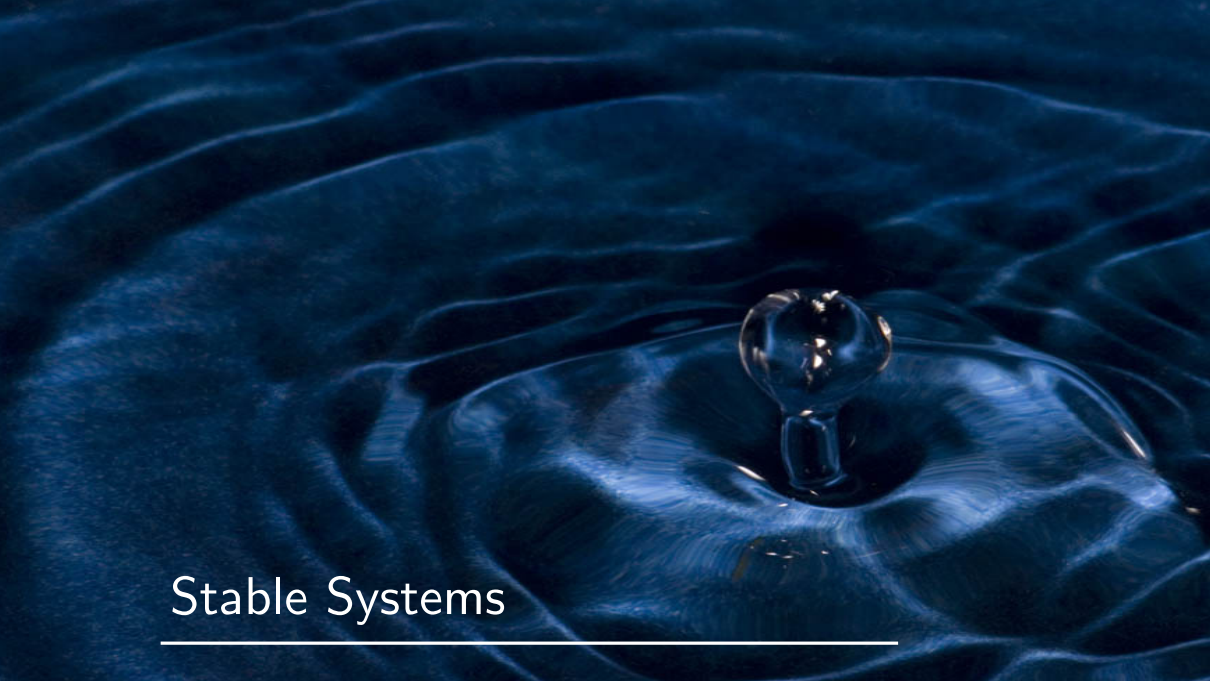
- Armed with this fact, we can implement infinite-length convolution using circular convolution
  1. Extract the D_x-sample support interval of x and zero pad so that the resulting signal x' is of length $D_x + D_h - 1$
  2. Perform the same operations on h to obtain h'
  3. Circularly convolve $x' \circledast h'$ to obtain y'

- **Fact:** The values of the signal y' will coincide with those of the infinite-length convolution $y = x * h$ within its support interval

- How does it work? The zero padding effectively converts circular shifts (finite-length signals) into regular shifts (infinite-length signals) (Easy to try out in Matlab!)

Summary

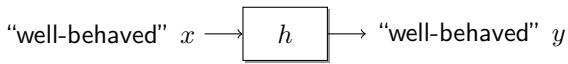
- Convolution has very special and beautiful properties
- Convolution is commutative
- Convolutions (LTI systems) can be connected in cascade and parallel
- An LTI system is causal if its impulse response is causal
- LTI systems are either FIR or IIR
- Can implement infinite-length convolution using circular convolution when the signals have finite duration (important later for "fast convolution" using the FFT)



Stable Systems

Stable Systems (1)

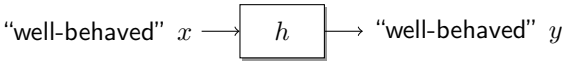
- With a **stable** system, a "well-behaved" input always produces a "well-behaved" output



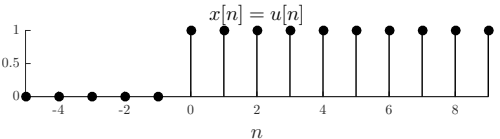
- Stability is essential to ensuring the proper and safe operation of myriad systems
  - Steering systems
  - Braking systems
  - Robotic navigation
  - Modern aircraft
  - International Space Station
  - Internet IP packet communication (TCP) ...

Stable Systems (2)

- With a **stable** system, a "well-behaved" input always produces a "well-behaved" output

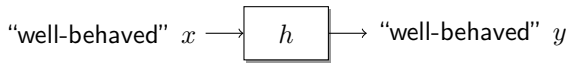


- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$
  Consider a step function input $x[n] = u[n]$



Well-Behaved Signals

- With a **stable** system, a "well-behaved" input always produces a "well-behaved" output



- How to measure how "well-behaved" a signal is? Different measures give different notions of stability

- One reasonable measure: A signal x is well behaved if it is **bounded** (recall that $\sup$ is like $\max$)

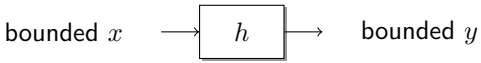
$$\|x\|_\infty \;=\; \sup_n |x[n]| \;<\; \infty$$

Bounded-Input Bounded-Output (BIBO) Stability

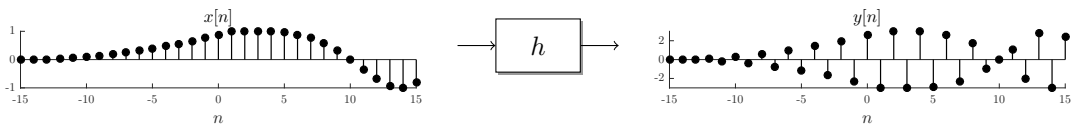
BIBO Stability (1)

DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



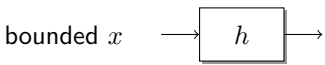
- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$,
  or that there exist constants $A, C < \infty$ such that $|x[n]| < A$ and $|y[n]| < C$ for all n



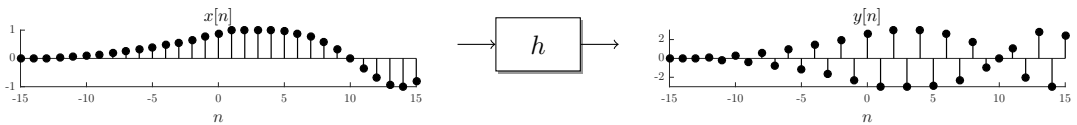
BIBO Stability (2)

DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$



- **Fact:** An LTI system with impulse response h is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

BIBO Stability – Sufficient Condition

- Prove that if $\|h\|_1 < \infty$ then the system is BIBO stable – for any input $\|x\|_\infty < \infty$ the output $\|y\|_\infty < \infty$
- Recall that $\|x\|_\infty < \infty$ means there exist a constant A such that $|x[n]| < A < \infty$ for all n
- Let $\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| = B < \infty$
- Compute a bound on $|y[n]|$ using the convolution of x and h and the bounds A and B

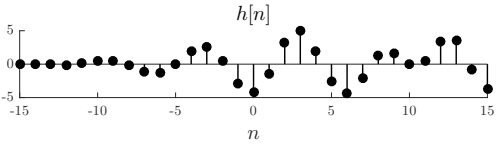
$$
\begin{aligned}
|y[n]| &= \left| \sum_{m=-\infty}^{\infty} h[n-m]\, x[m] \right| \leq \sum_{m=-\infty}^{\infty} |h[n-m]|\, |x[m]| \\
&< \sum_{m=-\infty}^{\infty} |h[n-m]|\, A = A \sum_{k=-\infty}^{\infty} |h[k]| = A\, B = C < \infty
\end{aligned}
$$

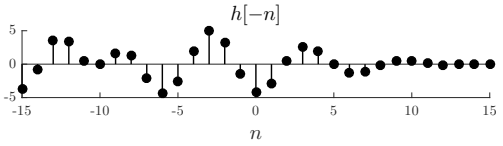
- Since $|y[n]| < C < \infty$ for all n, $\|y\|_\infty < \infty$ ✓

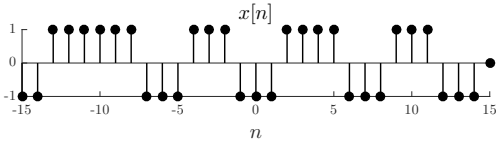
BIBO Stability – Necessary Condition (1)

- Prove that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
  - Assume that x and h are real-valued; the proof for complex-valued signals is nearly identical

- Given an impulse response h with $\|h\|_1 = \infty$, form the tricky special signal $x[n] = \mathrm{sgn}(h[-n])$
  - $x[n]$ is the $\pm$ sign of the time-reversed impulse response $h[-n]$
  - Note that x is bounded: $|x[n]| \leq 1$ for all n







BIBO Stability – Necessary Condition (2)

- We are proving that that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$

- Armed with the tricky special signal x, compute the output $y[n]$ at the time point $n = 0$

$$\begin{aligned} y[0] &= \sum_{m=-\infty}^{\infty} h[0-m]\,x[m] = \sum_{m=-\infty}^{\infty} h[-m]\,\mathrm{sgn}(h[-m]) \\ &= \sum_{m=-\infty}^{\infty} |h[-m]| = \sum_{k=-\infty}^{\infty} |h[k]| = \infty \end{aligned}$$

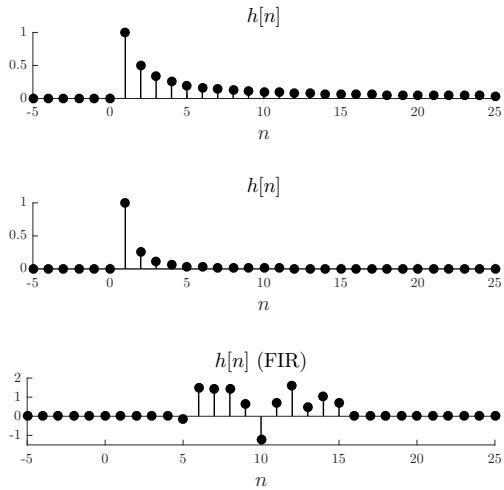
- So, even though x was bounded, y is not bounded; so system is not BIBO stable

BIBO System Examples (1)

- Absolute summability of the impulse response h determines whether an LTI systems is BIBO stable or not
- Example: $h[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

  $\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n}\right| = \infty \;\Rightarrow\;$ not BIBO
- Example: $h[n] = \begin{cases} \frac{1}{n^2} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

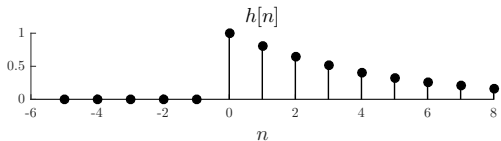
  $\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right| = \frac{\pi^2}{6} \;\Rightarrow\;$ BIBO
- Example: h FIR $\;\Rightarrow\;$ BIBO



BIBO System Examples (2)

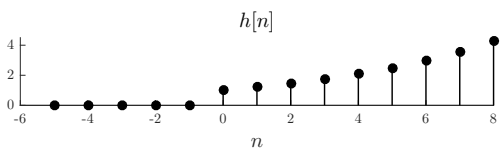
- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$
- Impulse response: $h[n] = \alpha^n\, u[n]$
- For $|\alpha| < 1$

$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} < \infty \;\Rightarrow$ BIBO



- For $|\alpha| > 1$

$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \infty \;\Rightarrow$ not BIBO

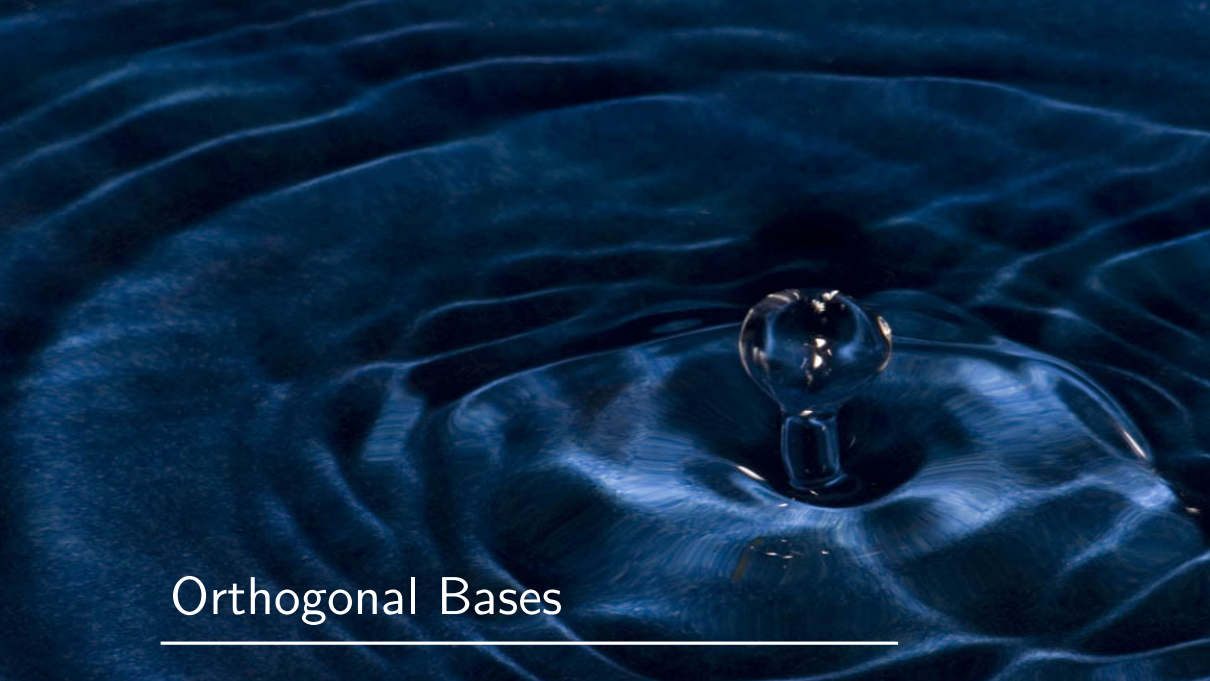


Summary

- Signal processing applications typically dictate that the system be **stable**, meaning that "well-behaved inputs" produce "well-behaved outputs"
- Measure "well-behavedness" of a signal using the ∞-norm (bounded signal)
- BIBO stability: bounded inputs always produce bounded outputs iff the impulse response h is such that $\|h\|_1 < \infty$
- When a system is not BIBO stable, all hope is not lost; unstable systems can often by **stabilized** using **feedback** (more on this later)

Orthogonal Bases and the DFT

Set E



Orthogonal Bases

Transforms and Orthogonal Bases

- We now turn back to linear algebra to understand **transforms**, which map signals between different "domains"
- Recall that signals can be interpreted as vectors in a vector space (linear algebra)
- We now review the concept of a **basis** for a vector space
- As we will see, different signal transforms (and "domains") correspond to different bases
- Caveat: This is not a course on linear algebra!

Basis

DEFINITION

A **basis** $\{b_k\}$ for a vector space V is a collection of vectors from V that are linearly independent and span V

- **Span:** All vectors in V can be represented as a linear combination of the basis vectors $\{b_k\}_{k=0}^{N-1}$

$$x \;=\; \sum_{k=0}^{N-1} \alpha_k \, b_k \;=\; \alpha_0 \, b_0 + \alpha_1 \, b_1 + \cdots + \alpha_{N-1} \, b_{N-1} \quad \forall \, x \in V$$

- **Linearly independent:** None of the basis vectors can be represented as a linear combination of the other basis vectors
- **Dimension of V:** The number of vectors in the basis (N in the above)
- **Fact:** The dimension of $\mathbb{R}^N$ and $\mathbb{C}^N$ equals N (we will focus on these spaces)

Basis Matrix

- Stack the basis vectors b_k as columns into the $N \times N$ **basis matrix**

$$\mathbf{B} \;=\; \left[b_0|b_1|\cdots|b_{N-1}\right]$$

- Stack the scalar weights α_k into an $N \times 1$ column vector

$$a = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$$

- We can now write a linear combination of basis elements as the matrix/vector product

$$x \;=\; \alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_{N-1} b_{N-1} \;=\; \sum_{k=0}^{N-1} \alpha_k b_k \;=\; \left[b_0|b_1|\cdots|b_{N-1}\right] \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \;=\; \mathbf{B}\, a$$

Orthogonal and Orthonormal Bases

DEFINITION

An **orthogonal basis** $\{b_k\}_{k=0}^{N-1}$ for a vector space V is a basis whose elements are mutually orthogonal

$$\langle b_k, b_l \rangle \;=\; 0, \quad k \neq l$$

DEFINITION

An **orthonormal basis** $\{b_k\}_{k=0}^{N-1}$ for a vector space V is a basis whose elements are mutually orthogonal and normalized (in the 2-norm)

$$\langle b_k, b_l \rangle \;=\; 0, \quad k \neq l$$

$$\|b_k\|_2 \;=\; 1$$

Example: Orthogonal and Orthonormal Bases in $\mathbb{R}^2$

$$b_0 = \begin{bmatrix} \quad \\ \quad \end{bmatrix}, \quad b_1 = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Inverse of a Matrix

DEFINITION

The **inverse** of a square matrix $\mathbf{A}$ is a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix of the same size as $\mathbf{A}$

- If $\mathbf{B}$ is a basis matrix and $x = \mathbf{B}a$, then $a = \mathbf{B}^{-1}x$

Inverse of an Orthonormal Basis Matrix

- When the basis matrix $\mathbf{B}$ contains an **orthonormal basis** $\{b_k\}_{k=0}^{N-1}$, its inverse $\mathbf{B}^{-1}$ is trivial to calculate

- **Fact:** $\mathbf{B}^{-1} = \mathbf{B}^H$ (recall ^{H} is complex conjugate transpose)

- Terminology: $\mathbf{B}$ is a **unitary** matrix (aka complex-valued orthogonal)

- To prove, write out $\mathbf{B}^H\mathbf{B}$ and use the fact that the columns of $\mathbf{B}$ are orthonormal

$$\mathbf{B}^H\mathbf{B} = \begin{bmatrix} b_0^H \\ b_1^H \\ \vdots \\ b_{N-1}^H \end{bmatrix} \begin{bmatrix} b_0 | b_1 | \cdots | b_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Signal Representation by Orthonormal Basis

- Given an **orthonormal basis** $\{b_k\}_{k=0}^{N-1}$ and orthonormal basis matrix $\mathbf{B}$, we have the following **signal representation** for any signal x

$$x \;=\; \mathbf{B}\,a \;=\; \sum_{k=0}^{N-1} \alpha_k\, b_k \qquad \text{(synthesis)}$$

$$a \;=\; \mathbf{B}^H x \quad \text{or, each} \quad \alpha_k \;=\; \langle x, b_k \rangle \qquad \text{(analysis)}$$

- **Synthesis:** Build up the signal x as a linear combination of the basis elements b_k weighted by the weights α_k

- **Analysis:** Compute the weights α_k such that the synthesis produces x; the weight α_k measures the similarity between x and the basis element b_k

Summary

- **Orthonormal bases** make life easy

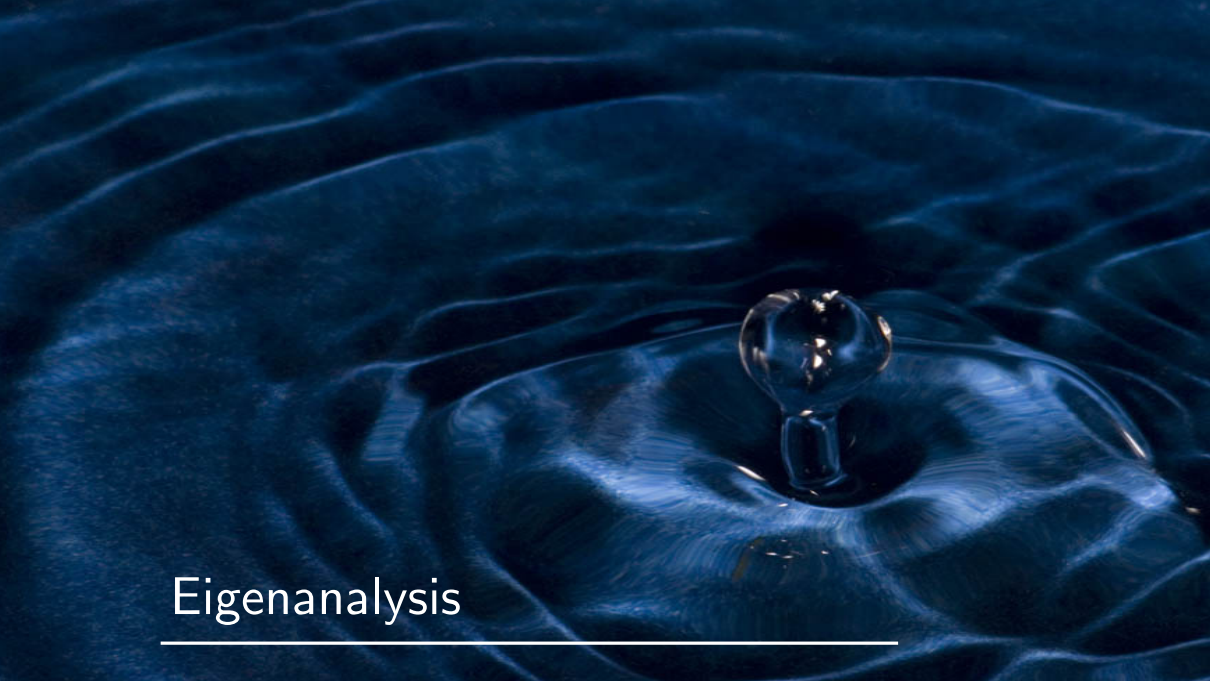
- Given an **orthonormal basis** $\{b_k\}_{k=0}^{N-1}$ and orthonormal basis matrix $\mathbf{B}$, we have the following **signal representation** for any signal x

$$x \;=\; \mathbf{B}\,a \;=\; \sum_{k=0}^{N-1} \alpha_k\, b_k \qquad \text{(synthesis)}$$

$$a \;=\; \mathbf{B}^H x \quad \text{or, each} \quad \alpha_k \;=\; \langle x, b_k \rangle \qquad \text{(analysis)}$$

- In signal processing, we say that the vector a is the **transform** of the signal x with respect to the orthonormal basis $\{b_k\}_{k=0}^{N-1}$

- Clearly the transform a contains all of the information in the signal x (and vice versa)



Eigenanalysis

Eigenanalysis

- We continue borrowing from linear algebra by recalling the **eigenvectors** and **eigenvalues** of a matrix
- Applying this point of view to circulant matrices (LTI systems for finite-length signals) will lead to an amazing result that ties together many threads of thought
- Caveat: This is not a course on linear algebra!

Eigenvectors and Eigenvalues

DEFINITION

Given a square matrix $\mathbf{A}$, the vector v is an **eigenvector** with **eigenvalue** λ if

$$\mathbf{A}\,v \;=\; \lambda\,v$$

- Geometric intuition: Multiplying an eigenvector v by the matrix $\mathbf{A}$ does not change its direction; it changes only its strength by the factor $\lambda \in \mathbb{C}$
- Example in $\mathbb{R}^2$:

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda = 2$$

$$\mathbf{A}v = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2v$$

Eigendecomposition

- An $N \times N$ matrix $\mathbf{A}$ has N eigenvectors and N eigenvalues (not necessarily distinct, though)
- Stack the N eigenvectors $\{v_m\}_{m=0}^{N-1}$ as columns into an $N \times N$ matrix

$$\mathbf{V} = \left[v_0 | v_1 | \cdots | v_{N-1}\right]$$

- Place the N eigenvalues $\{\lambda_m\}_{m=0}^{N-1}$ on the diagonal of an $N \times N$ diagonal matrix

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{N-1} \end{bmatrix}$$

- Then we can write

$$\mathbf{AV} = \mathbf{V\Lambda}$$

Diagonalization

- Recall the eigendecomposition of a matrix $\mathbf{A}$

$$\mathbf{AV} = \mathbf{V\Lambda}$$

- When the eigenvector matrix $\mathbf{V}$ is **invertible**, we can multiply both sides of the eigendecomposition on the left by $\mathbf{V}^{-1}$ to obtain

$$\mathbf{V}^{-1}\mathbf{AV} = \mathbf{V}^{-1}\mathbf{V\Lambda} = \mathbf{I\Lambda} = \mathbf{\Lambda}$$

- We say that the eigenvector matrix $\mathbf{V}$ **diagonalizes** the matrix $\mathbf{A}$

$$\mathbf{V}^{-1}\mathbf{AV} = \mathbf{\Lambda}$$

- Much easier to multiply a vector by $\mathbf{\Lambda}$ than by $\mathbf{A}$! (We simply scale each entry)

- Clearly $\mathbf{V}$ and $\mathbf{\Lambda}$ contain all of the information in $\mathbf{A}$ (and vice versa)

Aside: Diagonalization and Normal Matrices

- A matrix $\mathbf{A}$ is **normal** if $AA^T = A^TA$
- Given a normal matrix $\mathbf{A}$, if its eigenvector matrix $\mathbf{V}$ is invertible, then it is also **orthogonal**
- **Circulant matrices** are normal
- Recall that the system matrix $\mathbf{H}$ of a finite-length LTI system is circulant, and therefore also normal.
- Thus the eigenvector matrix $\mathbf{V}$ of a circulent matrix $\mathbf{H}$ is orthogonal

Summary

- Multiplying an eigenvector v by the matrix $\mathbf{A}$ does not change its direction; it changes only its strength by the factor λ

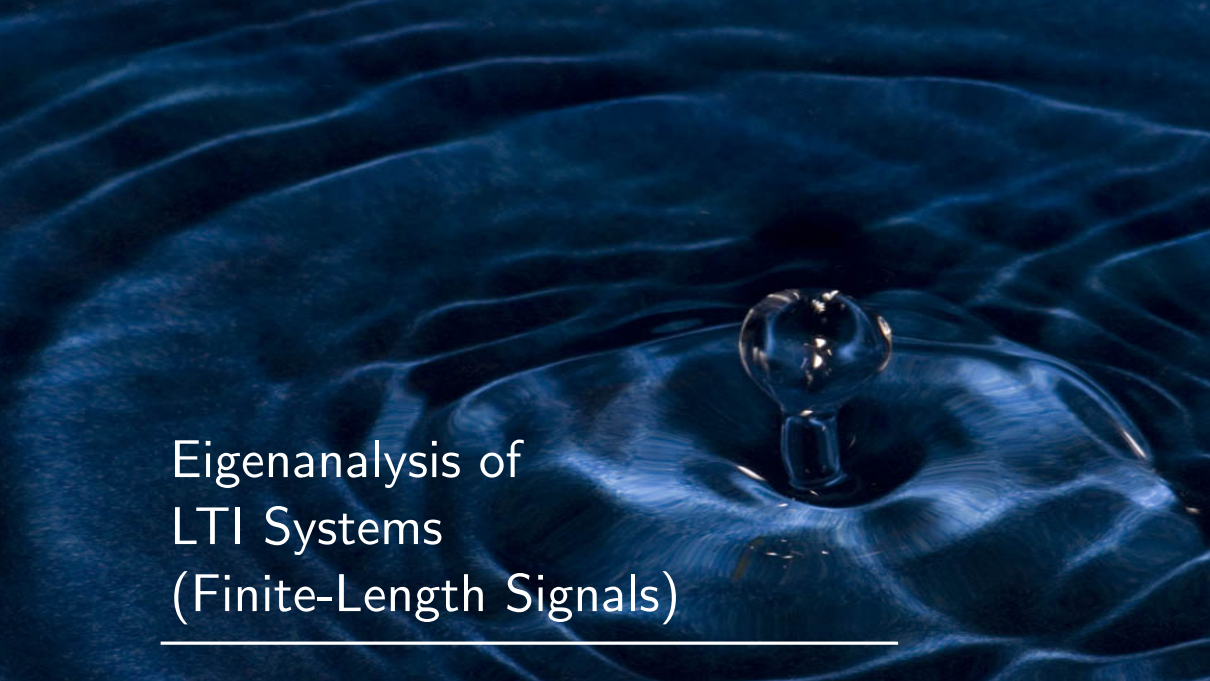
- The eigenvectors/values contain all of the information in the matrix $\mathbf{A}$ (and vice versa)

- Diagonalization by eigendecomposition

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}$$

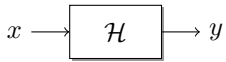
or, equivalently,

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$$



Eigenanalysis of LTI Systems (Finite-Length Signals)

LTI Systems for Finite-Length Signals



$$y \;=\; \mathbf{H}x$$

- For length-N signals, $\mathbf{H}$ is an $N \times N$ **circulant matrix** with entries

$$[\mathbf{H}]_{n,m} \;=\; h[(n-m)_N]$$

  where h is the **impulse response**

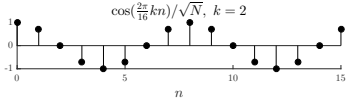
- **Goal:** Calculate the **eigenvectors** and **eigenvalues** of $\mathbf{H}$

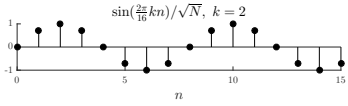
- Eigenvectors v are input signals that emerge at the system output unchanged (except for a scaling by the eigenvalue λ) and so are somehow "fundamental" to the system

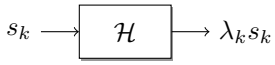
Eigenvectors of LTI Systems

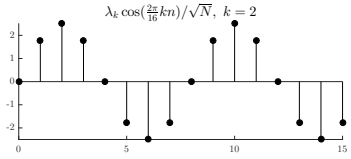
- **Fact:** The eigenvectors of a circulant matrix (LTI system) are the complex **harmonic sinusoids**

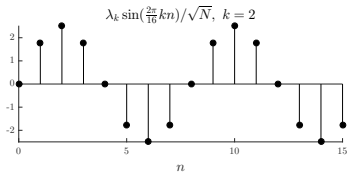
$$s_k[n] \;=\; \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}} \;=\; \frac{1}{\sqrt{N}}\left(\cos\left(\frac{2\pi}{N}kn\right) + j\sin\left(\frac{2\pi}{N}kn\right)\right), \qquad 0 \le n,k \le N-1$$



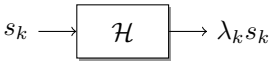








Harmonic Sinusoids are Eigenvectors of LTI Systems



- Prove that harmonic sinusoids are the eigenvectors of LTI systems simply by computing the circular convolution with input s_k and applying the periodicity of the harmonic sinusoids

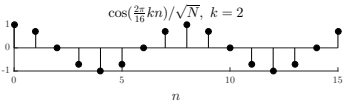
$$
\begin{aligned}
s_k[n] \circledast h[n] &= \sum_{m=0}^{N-1} s_k[(n-m)_N]\, h[m] = \sum_{m=0}^{N-1} \frac{e^{j\frac{2\pi}{N}k(n-m)_N}}{\sqrt{N}}\, h[m] \\
&= \sum_{m=0}^{N-1} \frac{e^{j\frac{2\pi}{N}k(n-m)}}{\sqrt{N}}\, h[m] = \sum_{m=0}^{N-1} \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}} e^{-j\frac{2\pi}{N}km}\, h[m] \\
&= \left(\sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}km}\, h[m] \right) \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}} = \lambda_k\, s_k[n]
\end{aligned}
$$

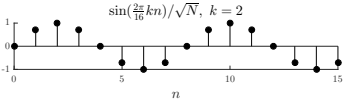
Eigenvalues of LTI Systems

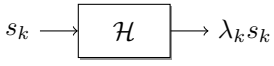
- The eigenvalue $\lambda_k \in \mathbb{C}$ corresponding to the sinusoid eigenvector s_k is called the **frequency response** at frequency k since it measures how the system "responds" to s_k

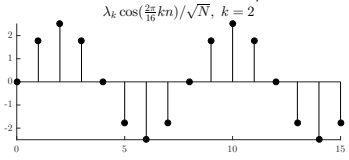
$$\lambda_k \;=\; \sum_{n=0}^{N-1} h[n]\, e^{-j\frac{2\pi}{N}kn} \;=\; \langle h, s_k \rangle \;=\; H_u[k] \quad \text{(unnormalized DFT)}$$

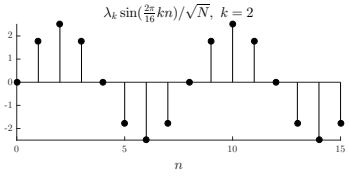
- Recall properties of the **inner product**: λ_k grows/shrinks as h and s_k become more/less similar











Eigenvector Matrix of Harmonic Sinusoids

- Stack the N normalized harmonic sinusoid $\{s_k\}_{k=0}^{N-1}$ as columns into an $N \times N$ complex orthonormal basis matrix

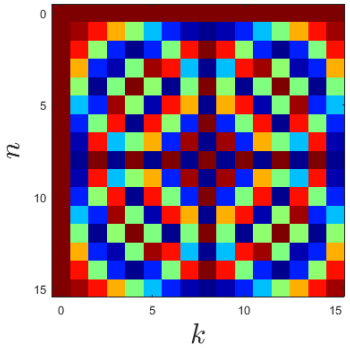
$$\mathbf{S} = \left[s_0 | s_1 | \cdots | s_{N-1}\right]$$

- The row-n, column-k entries of $\mathbf{S}$ have a very simple structure: $[\mathbf{S}]_{n,k} = e^{j\frac{2\pi}{N}kn}/\sqrt{N}$

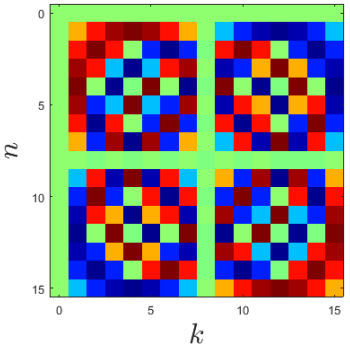
- Example: Eigenvector matrix for $N = 16$

- Note the symmetries:
  $[\mathbf{S}]_{n,k} = [\mathbf{S}]_{k,n}$
  or $\mathbf{S} = \mathbf{S}^T$
  (same sinusoids on rows/columns)

real part: $\cos(\frac{2\pi}{N}kn)/\sqrt{N}$



imaginary part: $\sin(\frac{2\pi}{N}kn)/\sqrt{N}$



Diagonal Matrix of Eigenvalues

- The eigenvalues are the frequency response (unnormalized DFT of the impulse response)

$$\lambda_k \;=\; \sum_{n=0}^{N-1} h[n]\, e^{-j\frac{2\pi}{N}kn} \;=\; \langle h, s_k \rangle \;=\; H_u[k] \;\; \text{(unnormalized DFT)}$$

- Place the N eigenvalues $\{\lambda_k\}_{k=0}^{N-1}$ on the diagonal of an $N \times N$ matrix

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{N-1} \end{bmatrix} = \begin{bmatrix} H_u[0] & & & \\ & H_u[1] & & \\ & & \ddots & \\ & & & H_u[N-1] \end{bmatrix}$$

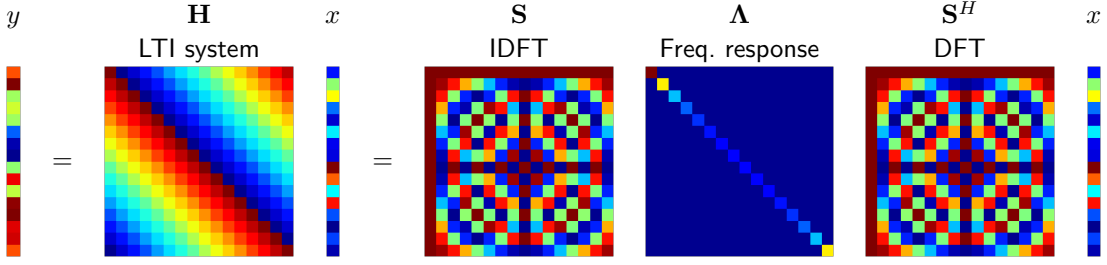
Eigendecomposition and Diagonalization of an LTI System

- Given the
  - circulant LTI system matrix $\mathbf{H}$
  - Fixed matrix of harmonic sinusoid eigenvectors $\mathbf{S}$ (corresponds to DFT/IDFT)
  - Diagonal matrix of eigenvalues $\mathbf{\Lambda}$ (frequency response, changes with $\mathbf{H}$)

  we can write

$$\mathbf{H} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^H$$

- Example for $N = 16$ (Note: Only plotting real part of $\mathbf{S}$, $\mathbf{S}^H$, and $\mathbf{\Lambda}$)



Summary

- Harmonic sinusoids are the eigenfunctions of LTI systems for finite-length signals (circulant matrices)
- Therefore, the discrete Fourier transform (DFT) is the natural tool for studying LTI systems for finite-length signals
- Frequency response $H[k]$ equals the unnormalized DFT of the impulse response $h[n]$
- Diagonalization by eigendecomposition implies

$$\mathbf{H} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^H$$

Discrete Fourier Transform (DFT)

Discrete Fourier Transform

- Another cornerstone of this course in particular and signal processing in general
- Jean Baptiste Joseph Fourier (21 March 1768 – 16 May 1830) had the radical idea of proposing that "all" signals could be represented as a linear combination of sinusoids

- Amazingly, it's true (at least in $\mathbb{C}^N$)!
- Suggestion: Re-watch the lectures on Sinusoids from Week 1

Recall: Signal Representation by Orthonormal Basis

- Given an **orthonormal basis** $\{b_k\}_{k=0}^{N-1}$ for $\mathbb{C}^N$ and orthonormal basis matrix $\mathbf{B}$, we have the following **signal representation** for any signal x

$$x \;=\; \mathbf{B}\,a \;=\; \sum_{k=0}^{N-1} \alpha_k\, b_k \qquad \text{(synthesis)}$$

$$a \;=\; \mathbf{B}^H x \quad \text{or} \quad \alpha_k \;=\; \langle x, b_k \rangle \qquad \text{(analysis)}$$

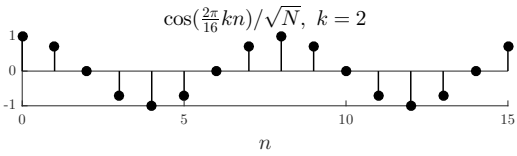
- **Synthesis:** Build up the signal x as a linear combination of the basis elements b_k weighted by the weights α_k

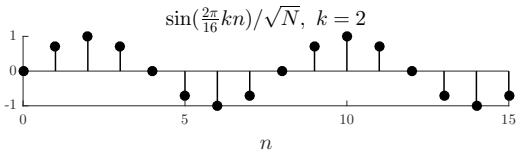
- **Analysis:** Compute the weights α_k such that the synthesis produces x; the weight α_k measures the similarity between x and the basis element b_k

Harmonic Sinusoids are an Orthonormal Basis

- Recall the length-N normalized complex **harmonic sinusoids** (normalized!)

$$s_k[n] \;=\; \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}} \;=\; \frac{1}{\sqrt{N}}\left(\cos\left(\frac{2\pi}{N}kn\right) + j\sin\left(\frac{2\pi}{N}kn\right)\right), \qquad 0 \le n, k \le N-1$$





- Recall that harmonic sinusoids are **mutually orthogonal** and **normalized**

$$\langle s_k, s_l \rangle \;=\; 0, \quad k \neq l, \qquad\qquad \|s_k\|_2 \;=\; 1$$

- It is easy to show that N orthonormal vectors in an N-dimensional must be an orthonormal basis

Orthonormal Basis Matrix of Harmonic Sinusoids

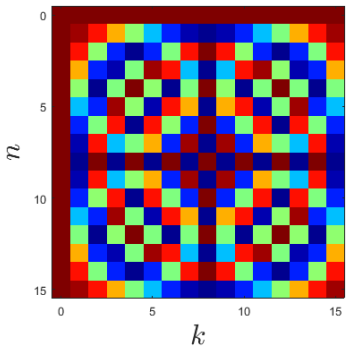
- Stack the N normalized harmonic sinusoid $\{s_k\}_{k=0}^{N-1}$ as columns into an $N \times N$ complex matrix

$$\mathbf{S} = \left[s_0 | s_1 | \cdots | s_{N-1} \right]$$

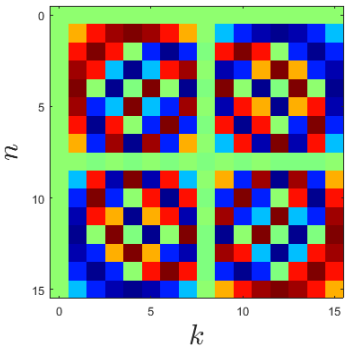
- The row-n, column-k entries of $\mathbf{S}$ have a very simple structure: $[\mathbf{S}]_{n,k} = e^{j\frac{2\pi}{N}kn}/\sqrt{N}$

- Example: Eigenvector matrix for $N = 16$

real part: $\cos(\frac{2\pi}{N}kn)/\sqrt{N}$



imaginary part: $\sin(\frac{2\pi}{N}kn)/\sqrt{N}$



- Note the symmetries:
  $[\mathbf{S}]_{n,k} = [\mathbf{S}]_{k,n}$
  or $\mathbf{S} = \mathbf{S}^T$
  (same sinusoids on rows/columns)

Inverse of Orthonormal Basis Matrix of Harmonic Sinusoids

- The row-n, column-k entries of $\mathbf{S}$ have a very simple structure

$$[\mathbf{S}]_{n,k} \;=\; \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

- $\mathbf{S}$ is **unitary**, and so its inverse matrix $\mathbf{S}^{-1} = \mathbf{S}^H$

- Entries of the inverse matrix

$$[\mathbf{S}^{-1}]_{n,k} \;=\; [\mathbf{S}^H]_{n,k} \;=\; \left(\frac{e^{j\frac{2\pi}{N}nk}}{\sqrt{N}}\right)^* \;=\; \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}} \;=\; ([\mathbf{S}]_{n,k})^*$$

- Thanks to the symmetry of $\mathbf{S}$, we have the interesting equalities: $\mathbf{S}^{-1} \;=\; \mathbf{S}^H \;=\; \mathbf{S}^*$

Signal Representation by Harmonic Sinusoids

- Given the normalized complex harmonic sinusoids $\{s_k\}_{k=0}^{N-1}$ and the orthonormal basis matrix $\mathbf{S}$, we define the (normalized) **discrete Fourier transform** (DFT) for any signal $x \in \mathbb{C}^N$

- **Analysis (Forward Normalized DFT)**

$$\begin{aligned} X &= \mathbf{S}^H x \\ X[k] &= \langle x, s_k \rangle = \sum_{n=0}^{N-1} x[n] \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}} \end{aligned}$$

- **Synthesis (Inverse Normalized DFT)**

$$\begin{aligned} x &= \mathbf{S}X \\ x[n] &= \sum_{k=0}^{N-1} X[k] \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}} \end{aligned}$$

Interpretation: Signal Representation by Harmonic Sinusoids

- **Analysis (Forward DFT)**
  - Choose the DFT coefficients $X[k]$ such that the synthesis produces the signal x
  - The weight $X[k]$ measures the similarity between x and the harmonic sinusoid s_k
  - Therefore, $X[k]$ measures the "frequency content" of x at frequency k

$$X[k] \;=\; \langle x, s_k \rangle \;=\; \sum_{n=0}^{N-1} x[n] \, \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

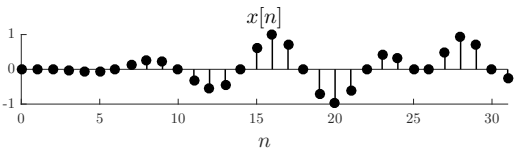
- **Synthesis (Inverse DFT)**
  - Build up the signal x as a linear combination of harmonic sinusoids s_k weighted by the DFT coefficients $X[k]$

$$x[n] \;=\; \sum_{k=0}^{N-1} X[k] \, \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

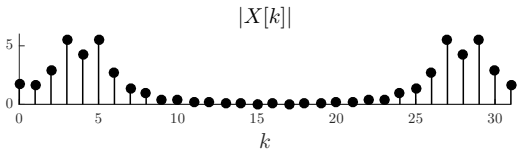
Example: Signal Representation by Harmonic Sinusoids

- **Analysis (Forward DFT)**
  - Choose the DFT coefficients $X[k]$ such that the synthesis produces the signal x
  - $X[k]$ measures the similarity between x and the harmonic sinusoid s_k
  - Therefore, $X[k]$ measures the "frequency content" of x at frequency k
  - Even if the signal x is real-valued, the DFT coefficients X will be complex, in general

$$X[k] \;=\; \langle x, s_k \rangle \;=\; \sum_{n=0}^{N-1} x[n] \, \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}}$$



time domain



frequency domain

The Unnormalized DFT

- Normalized forward and inverse DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

$$x[n] = \sum_{k=0}^{N-1} X[k] \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

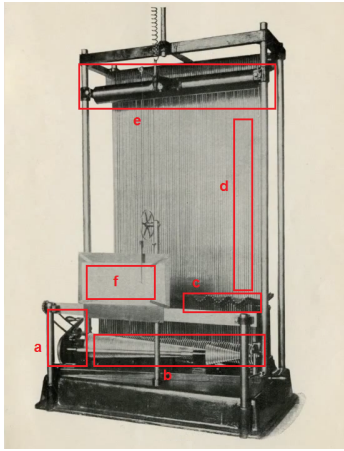
- **Unnormalized** forward and inverse DFT is more popular in practice (we will use both)

$$X_u[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

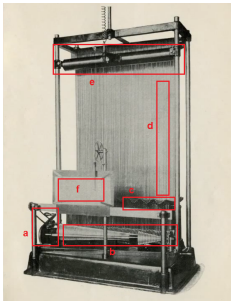
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_u[k]\, e^{j\frac{2\pi}{N}kn}$$

Aside: Physical Implementation of Fourier Analysis and Synthesis

- Before computers, Fourier analysis had to calculated done by hand
- Albert Michelson invented a machine that could physically perform such calculations

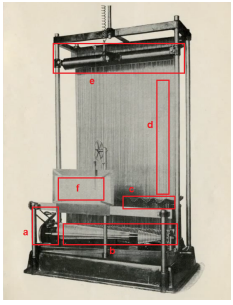


Aside: Physical Implementation of Fourier Analysis and Synthesis



- For synthesis, the a) crank turns a series of b) gears that spin at different rates, moving c) arms up and down in sinusoidal fashion
- d) Levers weight the various sinusoidal movements, which are summed with e) springs and a bar
- f) The result is plotted

Aside: Physical Implementation of Fourier Analysis and Synthesis



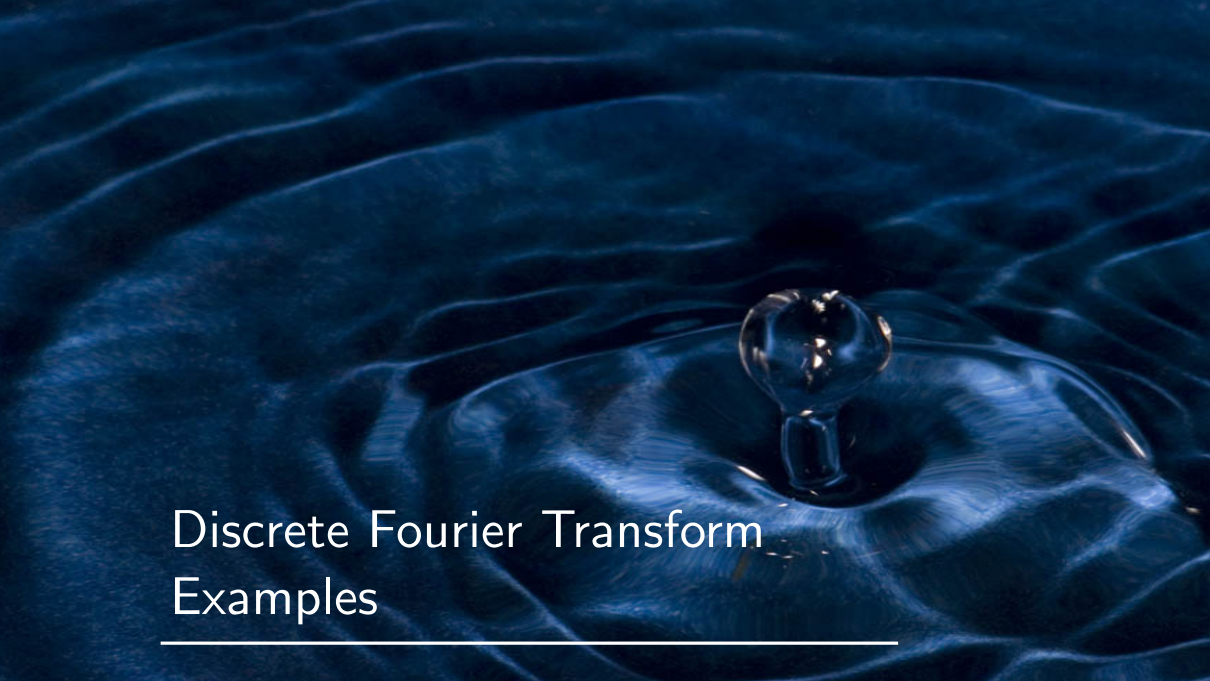
- For analysis, the d) levers are oriented into the shape of a single period of a signal
- Turning the crank amounts to taking the inner product of this shape with sinusoids (each turn makes a higher frequency sinusoid across the c) bars)
- The result is that the Fourier transform is then f) plotted
- To see videos of synthesis and analysis at work, see **http://www.engineerguy.com/fourier/**

Summary

- The discrete Fourier transform (DFT) is an orthonormal basis transformation based on the harmonic sinusoids

$$s_k[n] = \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

- The DFT maps signals from the "time domain" ($x[n]$) to the "frequency domain" ($X[k]$)
- The DFT coefficient $X[k]$ measures the similarity between the time signal x and the harmonic sinusoid s_k with frequency k
- The set of DFT coefficients X contains all of the information in the signal x (and vice versa)
- Do not confuse the normalized and unnormalized DFTs! The normalized DFT is more elegant, but the unnormalized DFT is much more popular in practice



Discrete Fourier Transform Examples

Discrete Fourier Transform

- Useful Matlab commands: `fft`, `fftshift`, `semilogy`. **Click here** to view a video demonstration.
- **Click here** to see other explanations and graphical representations of the DFT.

Discrete Fourier Transform Properties

Properties of the DFT

- **Normalized** forward and inverse DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

$$x[n] = \sum_{k=0}^{N-1} X[k] \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

- **Unnormalized** forward and inverse DFT is more popular in practice (we will use both)

$$X_u[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_u[k]\, e^{j\frac{2\pi}{N}kn}$$

DFT Pairs

- If $x[n]$ and $X[k]$ are such that

$$X[k] = \sum_{n=0}^{N-1} x[n] \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

$$x[n] = \sum_{k=0}^{N-1} X[k] \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

then we say they are a **DFT pair**

$$x[n] \overset{\text{DFT}}{\longleftrightarrow} X[k]$$

The DFT is Periodic

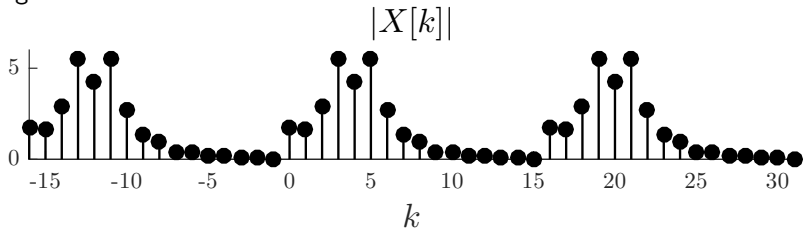
- The DFT is of finite length N, but it can also be interpreted as **periodic** with period N

$$X[k] \;=\; X[k+lN], \quad l \in \mathbb{Z}$$

- Proof

$$X[k+lN] \;=\; \sum_{n=0}^{N-1} x[n]\, \frac{e^{-j\frac{2\pi}{N}(k+lN)n}}{\sqrt{N}} \;=\; \sum_{n=0}^{N-1} x[n]\, \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}} e^{-j\frac{2\pi}{N}lNn} \;=\; X[k] \;\checkmark$$

- DFT of length $N = 16$

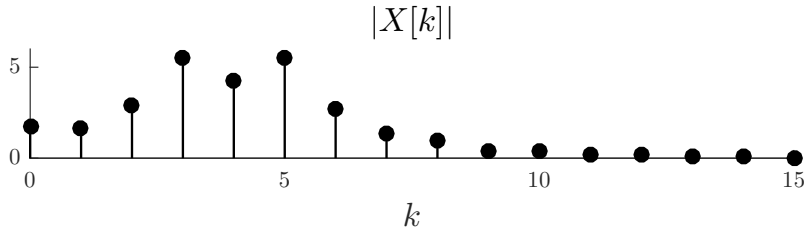


DFT Frequencies

$$X[k] \;=\; \langle x, s_k \rangle \;=\; \sum_{n=0}^{N-1} x[n] \, \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

- $X[k]$ measures the similarity between the time signal x and the harmonic sinusoid s_k
- Therefore, $X[k]$ measures the "frequency content" of x at frequency

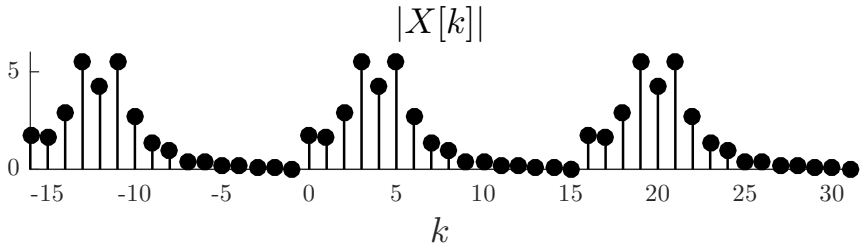
$$\omega_k \;=\; \frac{2\pi}{N} k$$



DFT Frequencies and Periodicity

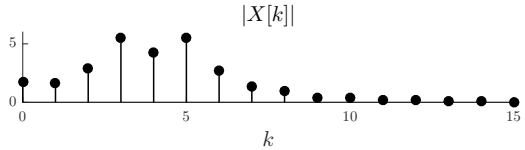
- Periodicity of DFT means we can treat frequencies mod N
- $X[k]$ measures the "frequency content" of x at frequency $\omega_k = \frac{2\pi}{N}(k)_N$
- Example: $X[N-1] = X[(-1)_N]$ measures the "frequency content" of x at the (same) frequencies

$$\omega_{N-1} = \frac{2\pi}{N}(N-1) = -\frac{2\pi}{N}$$

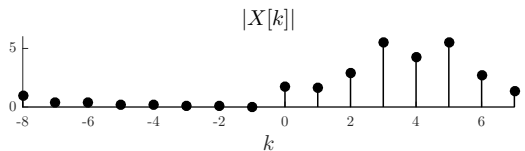


DFT Frequency Ranges

- Periodicity of DFT means every length-N interval of k carries the same information
- Typical interval 1: $0 \leq k \leq N-1$ corresponds to frequencies ω_k in the interval $0 \leq \omega < 2\pi$



- Typical interval 2: $-\frac{N}{2} \leq k \leq \frac{N}{2}-1$ corresponds to frequencies ω_k in the interval $-\pi \leq \omega < \pi$



The Inverse DFT is Periodic

$$x[n] \;=\; \sum_{k=0}^{N-1} X[k]\,\frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

- The time signal produced by the inverse DFT (synthesis) is **periodic** with period N

$$x[n] \;=\; x[n+mN], \quad m \in \mathbb{Z}$$

- Proof

$$x[n+mN] \;=\; \sum_{n=0}^{N-1} X[k]\,\frac{e^{j\frac{2\pi}{N}k(n+mN)}}{\sqrt{N}} \;=\; \sum_{n=0}^{N-1} X[k],\,\frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}e^{j\frac{2\pi}{N}kmN} \;=\; x[n] \;\checkmark$$

- This should not be surprising, since the harmonic sinusoids are periodic with period N

DFT and Circular Shift

- If $x[n]$ and $X[k]$ are a DFT pair then

$$x[(n-m)_N] \overset{\text{DFT}}{\longleftrightarrow} e^{-j\frac{2\pi}{N}km} X[k]$$

- Proof: Use the change of variables $r = (n-m)_N$

$$\sum_{n=0}^{N-1} x[(n-m)_N] \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}} = \sum_{r=0}^{N-1} x[r] \frac{e^{-j\frac{2\pi}{N}k(r+m)}}{\sqrt{N}} = e^{-j\frac{2\pi}{N}km} \sum_{r=0}^{N-1} x[r] \frac{e^{-j\frac{2\pi}{N}kr}}{\sqrt{N}}$$

$$= e^{-j\frac{2\pi}{N}km} X[k] \quad \checkmark$$

DFT and Modulation

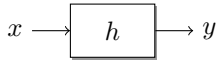
- If $x[n]$ and $X[k]$ are a DFT pair then

$$e^{j\frac{2\pi}{N}ln}\,x[n] \overset{\text{DFT}}{\longleftrightarrow} X[(k-l)_N]$$

- Proof:

$$\sum_{n=0}^{N-1} x[n]\,e^{j\frac{2\pi}{N}ln}\,\frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}} = \sum_{n=0}^{N-1} x[n]\,\frac{e^{-j\frac{2\pi}{N}(k-l)n}}{\sqrt{N}} = X[(k-l)_N] \quad \checkmark$$

DFT and Circular Convolution



$$y[n] \;=\; x[n] \circledast h[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m]$$

- If

$$x[n] \overset{\text{DFT}}{\longleftrightarrow} X_u[k], \qquad h[n] \overset{\text{DFT}}{\longleftrightarrow} H_u[k], \qquad y[n] \overset{\text{DFT}}{\longleftrightarrow} Y_u[k]$$

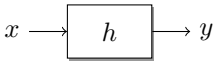
  then

$$Y_u[k] \;=\; H_u[k]\, X_u[k]$$

- Circular convolution in the time domain $\Leftrightarrow$ multiplication in the frequency domain

- Note: This is equivalent to $\mathbf{\Lambda} = \mathbf{S}^H \mathbf{H} \mathbf{S}$

DFT and Circular Convolution – Proof



$$y[n] \;=\; x[n] \circledast h[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m]$$

- Proof

$$Y_u[k] \;=\; \sum_{n=0}^{N-1} y[n]\, e^{-j\frac{2\pi}{N}kn} \;=\; \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} h[(n-m)_N]\, x[m] \right) e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{m=0}^{N-1} x[m] \left(\sum_{n=0}^{N-1} h[(n-m)_N] e^{-j\frac{2\pi}{N}kn} \right) \;=\; \sum_{m=0}^{N-1} x[m] \left(\sum_{r=0}^{N-1} h[r] e^{-j\frac{2\pi}{N}k(r+m)} \right)$$

$$= \left(\sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}km} \right) \left(\sum_{r=0}^{N-1} h[r] e^{-j\frac{2\pi}{N}kr} \right) \;=\; X_u[k]\, H_u[k] \quad \checkmark$$

The DFT is Linear

- It is trivial to show that if

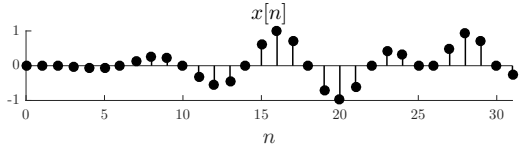
$$x_1[n] \overset{\text{DFT}}{\longleftrightarrow} X_1[k], \qquad x_2[n] \overset{\text{DFT}}{\longleftrightarrow} X_2[k]$$

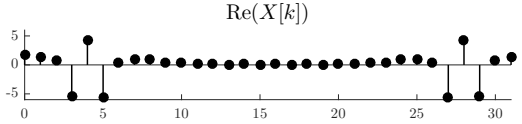
then

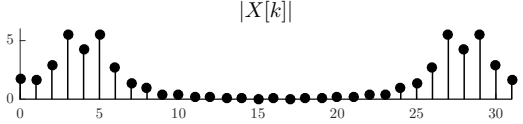
$$\alpha_1 x_1[n] + \alpha_2 x[2] \overset{\text{DFT}}{\longleftrightarrow} \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

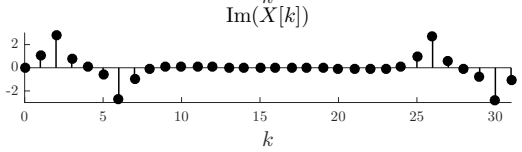
The DFT is Complex Valued

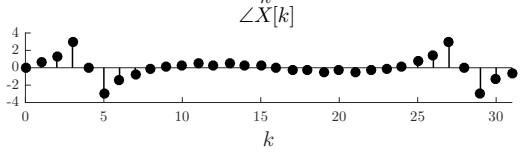
- Even if the signal $x[n]$ is real-valued, the DFT is complex-valued, in general











DFT Symmetry Properties (1)

- The harmonic sinusoid basis elements $s_k[n] = e^{j\frac{2\pi}{N}kn}$ of the DFT have symmetry properties:

$$\mathrm{Re}\left(e^{j\frac{2\pi}{N}kn}\right) = \cos\left(\frac{2\pi}{N}kn\right) \quad \text{(even function)}$$

$$\mathrm{Im}\left(e^{j\frac{2\pi}{N}kn}\right) = \sin\left(\frac{2\pi}{N}kn\right) \quad \text{(odd function)}$$

- These induce corresponding symmetry properties on $X[k]$ around the frequency $k = 0$

- **Even** signal/DFT

$$x[n] = x[(-n)_N], \qquad X[k] = X[(-k)_N]$$

- **Odd** signal/DFT

$$x[n] = -x[(-n)_N], \qquad X[k] = -X[(-k)_N]$$

DFT Symmetry Properties (2)

$x[n]$	$X[k]$	$\mathrm{Re}(X[k])$	$\mathrm{Im}(X[k])$	$\lvert X[k]\rvert$	$\angle X[k]$
real	$X[-k] = X[k]^*$	even	odd	even	odd
real & even	real & even	even	zero	even	
real & odd	imaginary & odd	zero	odd	even	
imaginary	$X[-k] = -X[k]^*$	odd	even	even	odd
imaginary & even	imaginary & even	zero	even	even	
imaginary & odd	real & odd	odd	zero	even	

DFT Symmetry Properties (3)

- Prove that if $x[n]$ is real, then $X[-k] = X[k]^*$

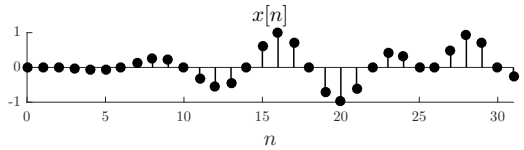
- Simply compute $X[k]^*$ and use the fact that $x[n]$ is real

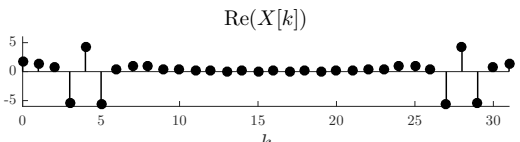
$$X[k]^* = \left(\sum_{n=0}^{N-1} x[n] \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}} \right)^* = \sum_{n=0}^{N-1} x[n]^* \frac{e^{+j\frac{2\pi}{N}kn}}{\sqrt{N}} = \sum_{n=0}^{N-1} x[n] \frac{e^{-j\frac{2\pi}{N}(-k)n}}{\sqrt{N}} = X[-k] \checkmark$$

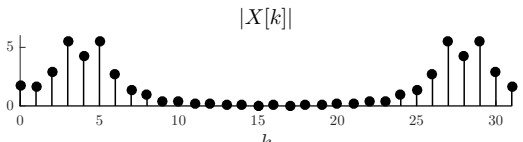
- Easy to continue on to prove that $\mathrm{Re}(X[-k]) = \mathrm{Re}(X[k])$ (that is, the real part of $X[k]$ is even) by taking the real part of both sides of the equation $X[-k] = X[k]^*$

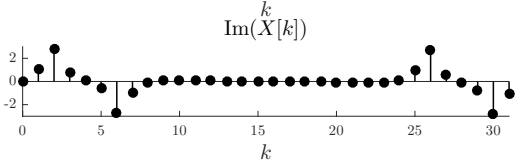
DFT Symmetry Properties (4)

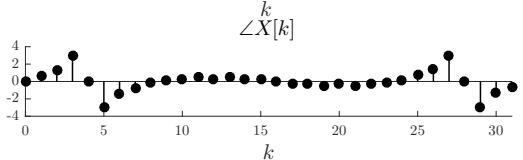
- Example: Real-valued signal $x[n]$





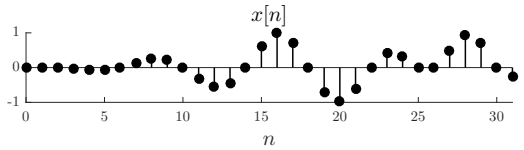


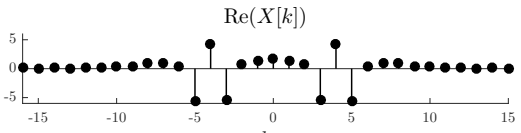


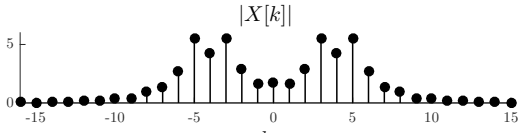


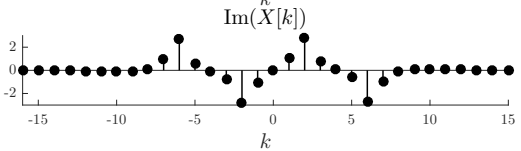
DFT Symmetry Properties (5)

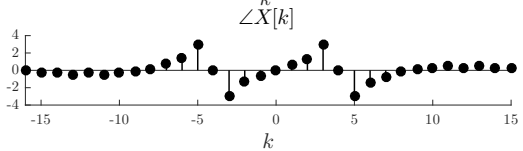
- Example: Real-valued signal $x[n]$, but plotting $X[k]$ using Matlab `fftshift` command











Duality of the DFT

- Note that the inverse and forward DFT formulas are identical except for conjugation of the harmonic sinusoids

$$X[k] = \sum_{n=0}^{N-1} x[n] \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

$$x[n] = \sum_{k=0}^{N-1} X[k] \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

- Thus, any DFT property that is true for $x[n]$ is also true for $X[-k]$

- Example: If $X[k]$ is real, then $\mathrm{Re}(x[n])$ is even and $\mathrm{Im}(x[n])$ is odd

Summary

- DFT and inverse DFT are periodic
- Useful to index $X[k]$ by $-\frac{N}{2} \leq k \leq \frac{N}{2} - 1$ (frequencies $-\pi \leq \frac{2\pi}{N} k < \pi$) as well as by $0 \leq k \leq N - 1$ (frequencies $0 \leq \frac{2\pi}{N} k < 2\pi$)
- Circular convolution in time becomes simple multiplication in frequency
- DFT has useful symmetry properties

Fast Fourier Transform (FFT)

Cost to Compute the Discrete Fourier Transform

- Recall the (unnormalized) DFT of the time signal $x[n]$

$$X_u[k] \;=\; \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad 0 \le k \le N-1$$

- What is the **computational cost** of the DFT?
- Number of Multiplies: Must multiply $x[n]\, e^{-j\frac{2\pi}{N}kn}$ for each value of

$$n = 0, 1, \ldots, N-1 \text{ and } k = 0, 1, \ldots, N-1 \;\Rightarrow\; N^2 \text{ total multiplies}$$

- Number of Additions: Must sum the N products $x[n]\, e^{-j\frac{2\pi}{N}kn}$ for each value of

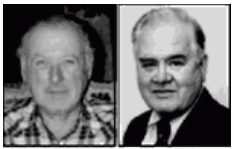
$$k = 0, 1, \ldots, N-1 \;\Rightarrow\; N(N-1) \approx N^2 \text{ total adds}$$

- Total computational cost of DFT: N^2 adds and N^2 multiplies – $O(N^2)$ **complexity**

Fast Fourier Transform

- $O(N^2)$ computational complexity is too high for many important applications; it is not uncommon to have $N = 10^7$ or more

- Important step forward in 1965: Cooley and Tukey "discovered" the fast Fourier transform (FFT), which lowers the computational complexity to $O(N \log N)$


James Cooley John Tukey

- Example: For $N = 10^7$

$$2N^2 = 2 \times 10^{14} \qquad 2N \log_2 N = 4.65 \times 10^8$$

- It turns out that Gauss invented the FFT in 1805 (Heideman, Johnson, Burrus, 1984)

Fast Fourier Transform

- There are many different kinds of FFTs; here we will study the simplest: the **radix-2, decimation-in-time** FFT

- Clearly we can use the same methods to speed up both the forward and inverse DFT (by duality); we will work with the forward DFT (and drop the subscript u for unnormalized)

$$X[k] \;=\; \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

- To keep the notation clean, define the **twiddle factor**: $W_N = e^{-j\frac{2\pi}{N}} \in \mathbb{C}$

$$X[k] \;=\; \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

Twiddle Factors are Periodic

- Note that the twiddle factors $W_N = e^{-j\frac{2\pi}{N}}$ are **periodic** in n and k

$$W_N^{kn} \;=\; W_N^{k(n+N)} \;=\; W_N^{(k+N)n}$$

- Proof

$$e^{-j\frac{2\pi}{N}kn} \;=\; e^{-j\frac{2\pi}{N}k(n+N)} \;=\; e^{-j\frac{2\pi}{N}(k+N)n}$$

FFT – Step 1

- In the radix-2, decimation-in-time FFT, the signal length N is a power of 2
- The FFT is a **divide and conquer** algorithm: We will split the length-N into two length-$N/2$ FFTs and then iterate; each split will save on computations
- We will work out the specific example of an $N = 8$ DFT, but the ideas extend to any power-of-two length
- **Step 1:** Break the signal $x[n]$ into two sub-signals:
  - **even samples**: $x[2n]$
  - **odd samples:** $x[2n+1], \quad n = 0, 1, \ldots, N/2 - 1$

$$X[k] \;=\; \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \;=\; \sum_{n=0}^{N/2-1} x[2n]\, W_N^{k(2n)} \;+\; \sum_{n=0}^{N/2-1} x[2n+1]\, W_N^{k(2n+1)}$$

FFT – Step 2

- **Step 2:** Reorganize the two sums into two length-$N/2$ DFTs

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} = \sum_{n=0}^{N/2-1} x[2n]\, W_N^{k(2n)} + \sum_{n=0}^{N/2-1} x[2n+1]\, W_N^{k(2n+1)}$$
$$= \sum_{n=0}^{N/2-1} x[2n]\, W_N^{2kn} + W_N^k \sum_{n=0}^{N/2-1} x[2n+1]\, W_N^{2kn}$$

- Note that $W_N^{2kn} = e^{-j\frac{2\pi}{N}2kn} = e^{-j\frac{2\pi}{N/2}kn} = W_{N/2}^{kn}$ and so we have ...

- Term 1 $= E[k] = \sum_{n=0}^{N/2-1} x[2n]\, W_{N/2}^{kn} = N/2$-point DFT of the even samples of $x[n]$

- Term 2 $= W_N^k\, O[k] = W_N^k \sum_{n=0}^{N/2-1} x[2n+1]\, W_{N/2}^{kn} = N/2$-point DFT of the odd samples of $x[n]$

FFT – Step 3

- **Step 3:** Not so fast! We need to evaluate

$$\begin{aligned} X[k] &= \sum_{n=0}^{N/2-1} x[2n]\, W_{N/2}^{kn} \;+\; W_N^k \sum_{n=0}^{N/2-1} x[2n+1]\, W_{N/2}^{kn} \\ &= E[k] \;+\; W_N^k\, O[k] \end{aligned}$$

for the entire range $k = 0, 1, \ldots, N-1$ and not just $k = 0, 1, \ldots, N/2-1$

- **Periodicity of the twiddle factors** implies that $E[k]$ and $O[k]$ are also periodic with period $N/2$

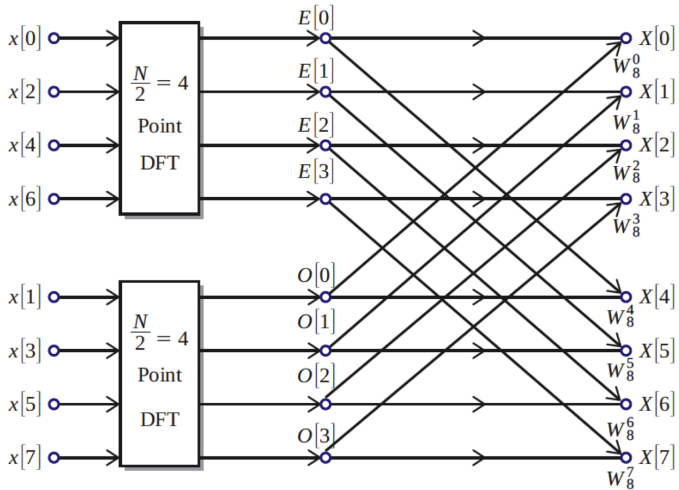
$$E[k+N/2] \;=\; \sum_{n=0}^{N/2-1} x[2n]\, W_{N/2}^{(k+N/2)n} \;=\; \sum_{n=0}^{N/2-1} x[2n]\, W_{N/2}^{kn} \;=\; E[k]$$

and similarly

$$O[k] \;=\; O[k+N/2]$$

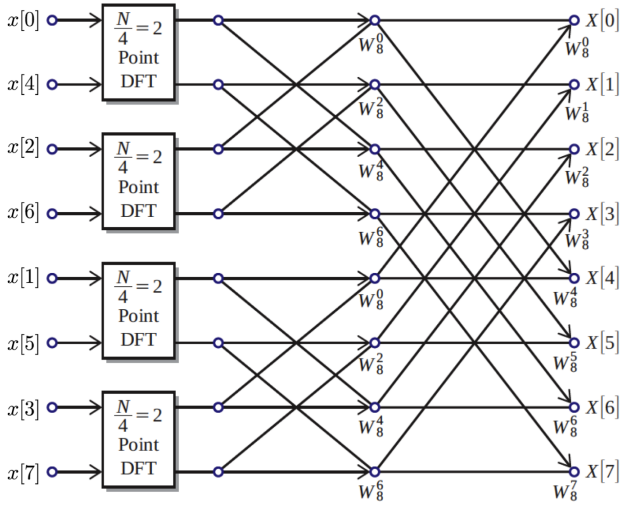
FFT – The Result

$$X[k] \;=\; E[k] + W_N^k\, O[k], \qquad k = 0, 1, \ldots, N-1$$



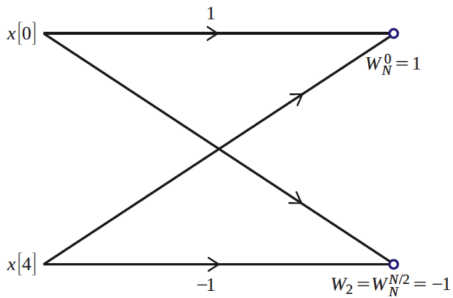
FFT – Iterate

- **Divide and Conquer!** Break the two length-$N/2$ DFTs into four length-$N/4$ DFTs

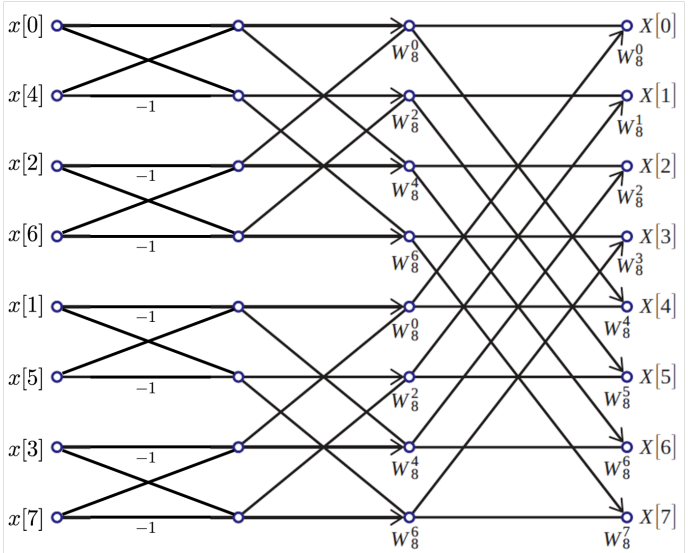


FFT – Divided and Conquered

- Iteration ends when we reach a length-2 DFT (here $N/4 = 2$)
- Length-2 DFT has a lovely **butterfly** structure



FFT of Length 8



Computational Savings

- **FFT:** Multiplies and adds required

$$2 \times 8 = 16 \text{ multiplies}$$

$$3 \times 8 = 24 \text{ adds}$$

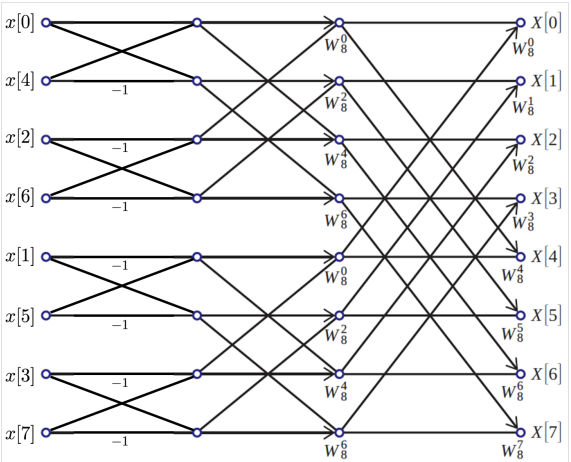
- **DFT:** Multiplies and adds required

$$8^2 = 64 \text{ multiplies}$$

$$8^2 - 8 = 56 \text{ adds}$$

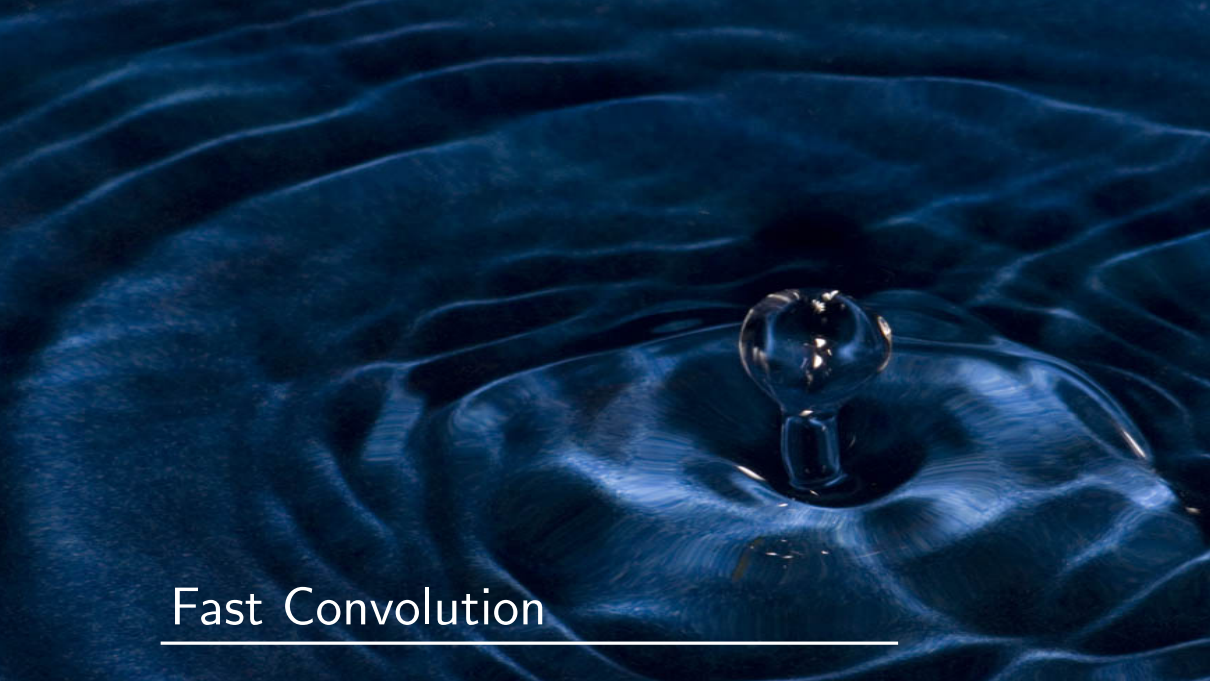
- In general, since a length-2^q FFT consists of q stages, the total number of multiplies and adds scales as

$$N \log_2 N \ll N^2$$



Summary

- The FFT has been called the "most important computational algorithm of our generation"
- The field of digital signal processing exploded after its introduction (1965)
- Why it works:
  - Symmetry and periodicity of sinusoids
  - Divide and conquer
- There are are many different kinds of FFTs for different lengths and different situations
- Rice University's resident FFT expert: Prof. C. Sidney Burrus



Fast Convolution

Cost to Compute a Circular Convolution

- Recall the circular convolution of two length-N time signals $x[n]$ and $h[n]$

$$y[n] \;=\; x[n] \circledast h[n] \;=\; \sum_{m=0}^{N-1} h[(n-m)_N]\, x[m], \qquad 0 \le n < N-1$$

- What is the **computational cost** of circular convolution?
- Number of Multiplies: Must multiply $h[(n-m)_N]\, x[n]$ for each value of

$$m = 0, 1, \ldots, N-1 \text{ and } n = 0, 1, \ldots, N-1 \;\Rightarrow\; N^2 \text{ total multiplies}$$

- Number of Additions: Must sum the N products $h[(n-m)_N]\, x[n]$ for each value of

$$n = 0, 1, \ldots, N-1 \;\Rightarrow\; N(N-1) \approx N^2 \text{ total adds}$$

- Total computational cost: N^2 adds and N^2 multiplies – $O(N^2)$ **complexity**

Circular Convolution via the FFT

- We can reduce the computational cost substantially if we move to the frequency domain using the DFT as computed by the FFT

- **Step 1:** Compute $H[k]$ and $X[k]$ of $h[n]$ and $x[n]$, respectively
  - Computational cost $= O(N \log N)$

- **Step 2:** Multiply $Y[k] = H[k] \times X[k]$
  - Computational cost $= O(N)$

- **Step 3:** Compute $y[n]$ via the inverse DFT of $Y[k]$
  - Computational cost $= O(N \log N)$

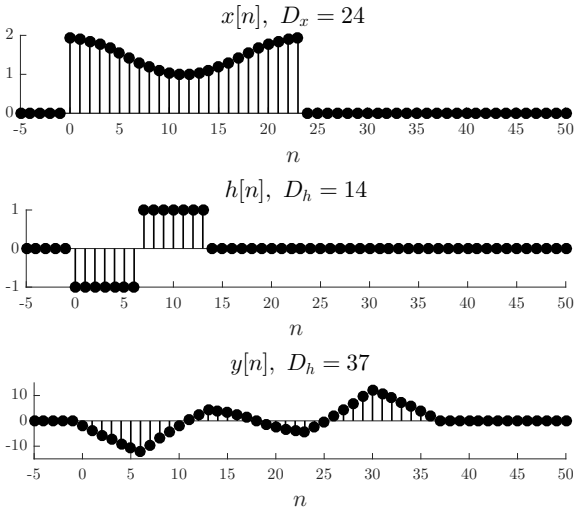
- Total computational cost: $O(N \log N) \ll N^2$

Extension to Fast Convolution of Infinite-Length, Finite-Duration Signals

- Applications tend to use (at least implicitly) infinite-length convolution more often than circular convolution
- Fortunately there is a clever way to trick a circular convolution into performing an infinite-length convolution
- Basic idea: zero pad the signals such that any circular wrap-around effects are zeroed out by the zero padding

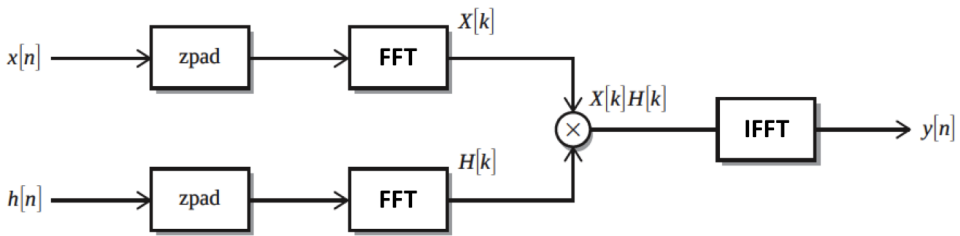
Duration of Convolution

- Recall that, if x has duration D_x samples and h has duration D_h samples, then the infinite-length convolution $y = x * h$ has duration at most $D_x + D_h - 1$ samples (proof by picture is simple)



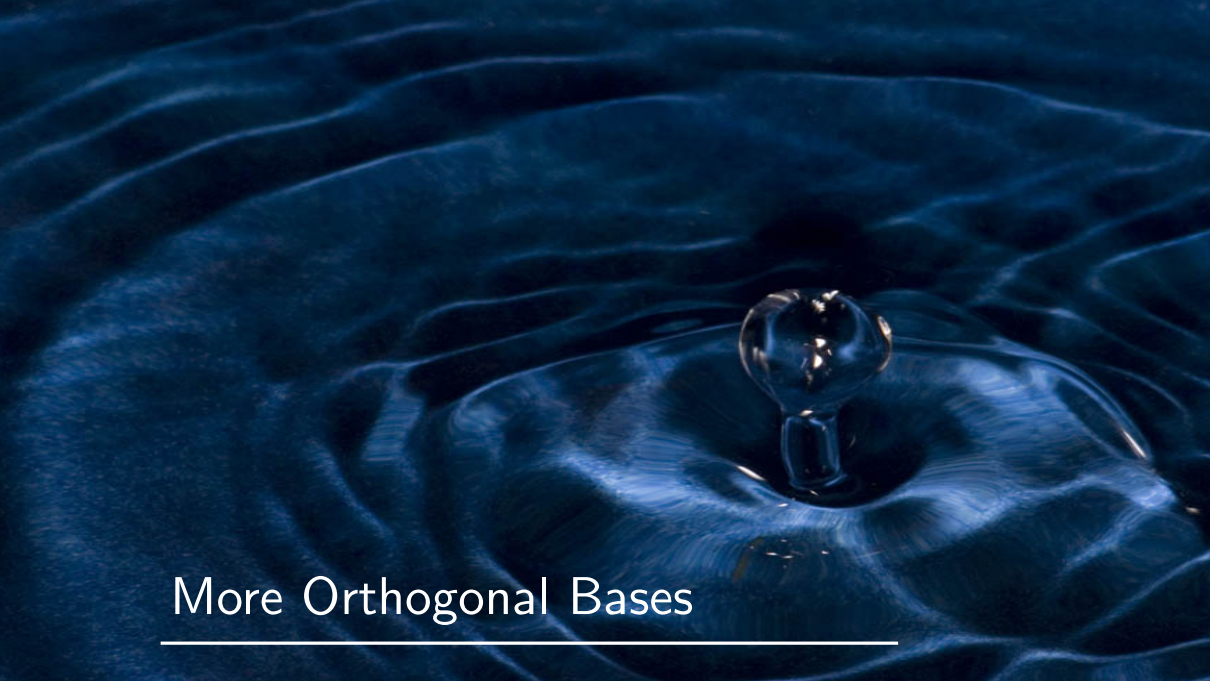
Extension to Fast Convolution of Infinite-Length, Finite-Duration Signals

- If x has duration D_x samples and h has duration D_h samples, then the infinite-length convolution $y = x * h$ has duration at most $D_x + D_h - 1$ samples
- If we zero pad both x and h to length $D_x + D_h - 1$ and compute their circular convolution $y' = x_{zp} \circledast h_{zp}$ ...
- Then the nonzero entries of the circular convolution y' will agree with those of the infinite-length convolution y



Summary

- Convolution computation can be expensive – $O(N^2)$ complexity
- Fast convolution is much faster – only $O(N \log N)$ complexity
- Can compute circular convolution by multiplying DFTs computing via FFT
- Can compute finite-duration, infinite-length convolution by zero padding before FFT



More Orthogonal Bases

Recall: Signal Representation by Orthonormal Basis

- Given an **orthonormal basis** $\{b_k\}_{k=0}^{N-1}$ for $\mathbb{C}^N$ and orthonormal basis matrix $\mathbf{B}$, we have the following **signal representation** for any signal x

$$x \;=\; \mathbf{B}\,a \;=\; \sum_{k=0}^{N-1} \alpha_k\, b_k \qquad \text{(synthesis)}$$

$$a \;=\; \mathbf{B}^H x \quad \text{or} \quad \alpha_k \;=\; \langle x, b_k \rangle \qquad \text{(analysis)}$$

- **Synthesis:** Build up the signal x as a linear combination of the basis elements b_k weighted by the weights α_k

- **Analysis:** Compute the weights α_k such that the synthesis produces x; the weight α_k measures the similarity between x and the basis element b_k

More Orthonormal Bases

- The DFT is the right transform to study LTI systems; the frequency domain arises naturally
- DFT is based on **complex-valued** basis elements (the harmonic sinusoids)
- **Challenge 1:** A signal $x \in \mathbb{R}^N$ has N complex DFT coefficients (real and imaginary parts of each DFT coefficient)
  - This is a problem in **compression** applications, where we would like approximate a smooth signal x by just a few DFT coefficients ($2\times$ redundancy)
- **Challenge 2:** Some signals are best represented neither in the time domain nor the frequency domain
  - For example, in a domain in between time and frequency ("time-frequency", like a musical score)
- Due to these and other challenges, there has been significant interest in developing additional orthonormal basis transformations beyond the DFT

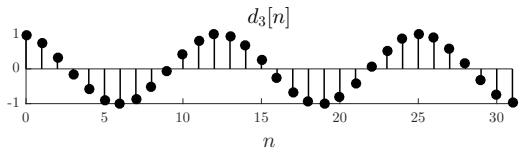
Discrete Cosine Transform (DCT)

- A DFT-like transform but using **real-valued basis functions** (`dct` in Matlab)
  - There are actually several different DCTs; here we present just one of them (the "DCT-II")
  - DCT is the transform inside JPEG image compression and MPEG video compression

- DCT Basis functions of length N (orthogonal)

$$d_k[n] \;=\; \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right], \quad 0 \le n,k \le N-1$$

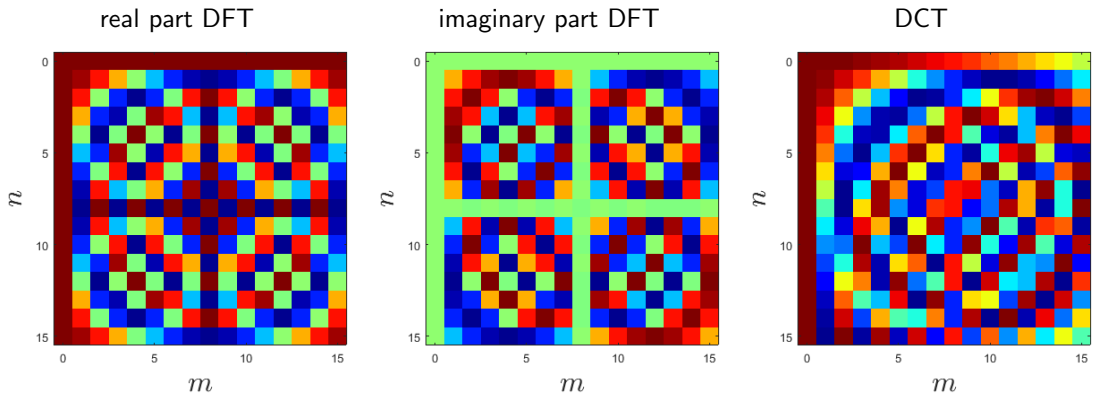
- Example: $N = 32,\ k = 3$



- Note: Not periodizable like the (complex) harmonic sinusoids of the DFT basis

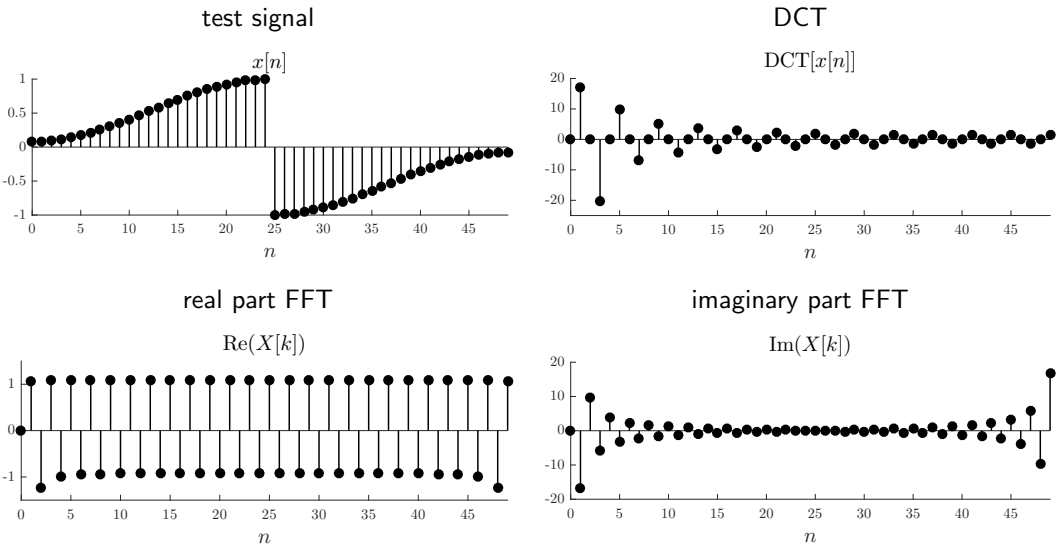
DCT Orthogonal Basis Matrix

- DCT basis matrix compared to the real/imaginary parts of the DFT basis matrix ($N = 16$)



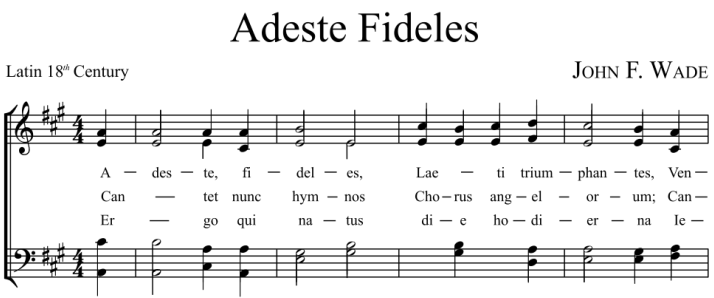
DCT vs. DFT for Compression

- DFT (real and imaginary parts) and DCT of a test signal



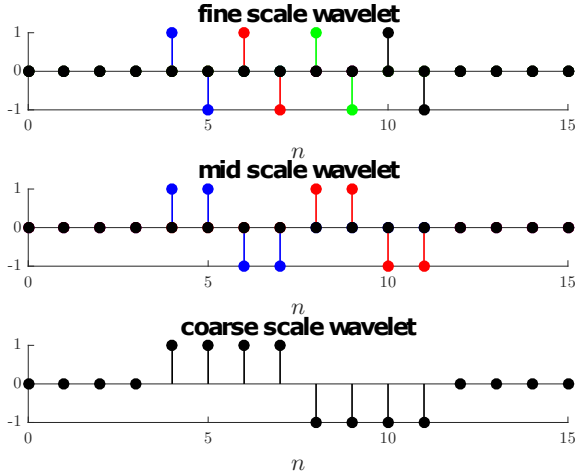
Between Time and Frequency

- Some signals are best represented neither in the time domain nor the frequency domain
- For example, many transient signals (audio, speech, images, etc.) are best represented in a domain in between time and frequency ("time-frequency", like a musical score)



Haar Wavelet Transform

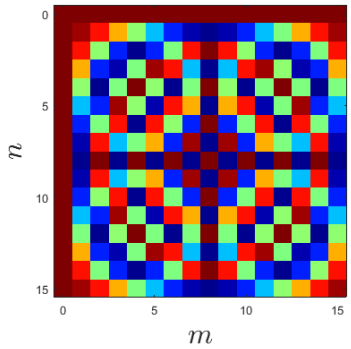
- Haar **wavelet transform** (1910): Key departure from DFT and DCT
  - Basis functions are **local** (short duration, "local waves")
  - Basis functions are **mulitscale** (many different durations) edge detectors (derivatives)



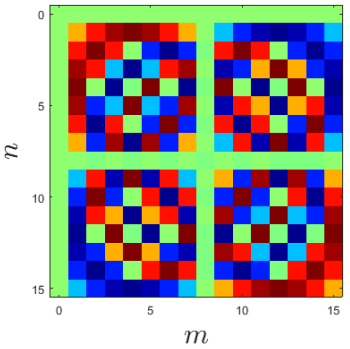
Haar Wavelet Transform Basis Matrix

- Wavelets are inside JPEG2000 image compression and many image analysis/processing systems
- Haar wavelet basis matrix compared to the real/imaginary parts of the DFT basis matrix ($N = 16$)

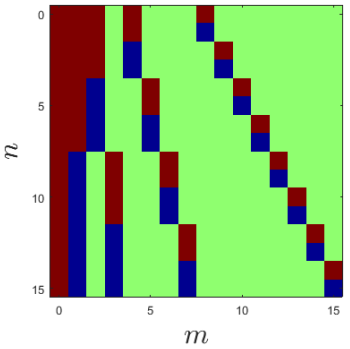
real part DFT



imaginary part DFT



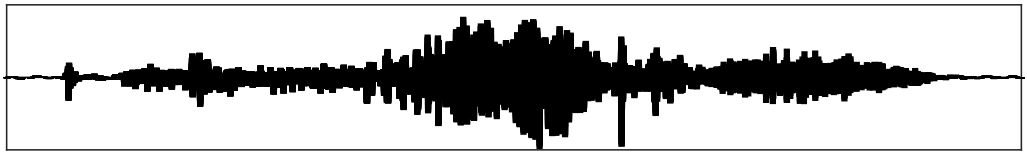
Haar wavelets



Short-Time Fourier Transform (1)

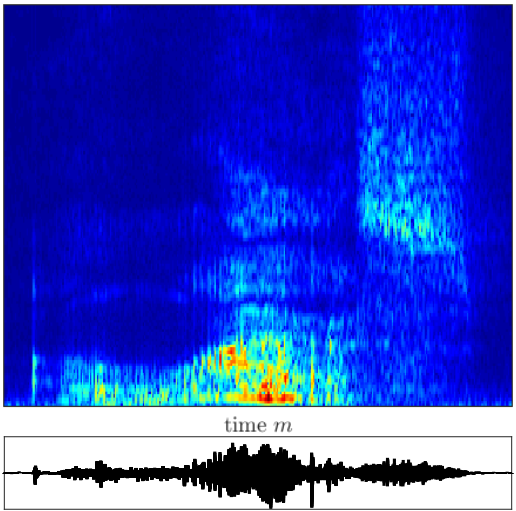
- STFT analyzes how a signal's frequency content changes over time – local Fourier analysis

- Given a signal $x[n]$ to analyze and a **window** $w[n]$
  - Window the signal with the window shifted to time m: $x[n]\, w[n-m]$
  - Compute the DFT of the windowed signal and stack as a column in an STFT matrix
  - Plot the STFT matrix as a 2D image (`imagesc` in Matlab, for example)



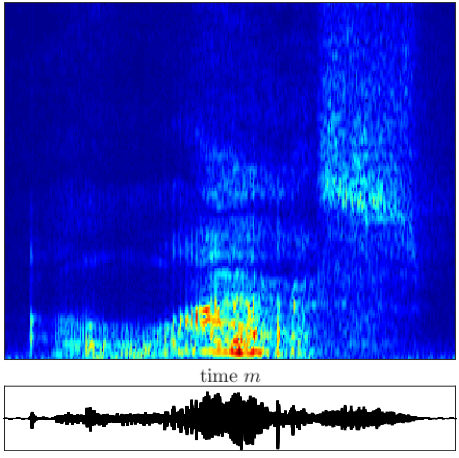
Short-Time Fourier Transform (2)

- STFT analyzes how a signal's frequency content changes over time – local Fourier analysis



Short-Time Fourier Transform (3)

- $|\text{STFT}|^2$ is called the **spectrogram** (`spectrogram` in Matlab)
- The STFT can be configured to be an orthonormal basis, but this is generally not done in practice



Summary

- The DFT is fundamental, especially for studying LTI systems
  - Only the DFT is defined in terms of basis functions that are eigenfunctions of LTI systems (harmonic sinusoids)
- But other orthogonal bases play important roles in diverse applications, especially signal analysis and compression
- Examples: DCT, wavelets, short-time Fourier transform, and beyond

The Discrete-Time Fourier Transform

Set F

Discrete Time Fourier Transform (DTFT)

Discrete Time Fourier Transform (DTFT)

- The DTFT is the Fourier transform of choice for analyzing infinite-length signals and systems
- Useful for conceptual, pencil-and-paper work, but not Matlab friendly (infinitely long vectors)
- Properties are very similar to the Discrete Fourier Transform (DFT) with a few caveats
- We will derive the DTFT as the limit of the DFT as the signal length $N \to \infty$

Recall: DFT (Unnormalized)

- **Analysis (Forward DFT)**
  - Choose the DFT coefficients $X[k]$ such that the synthesis produces the signal x
  - The weight $X[k]$ measures the similarity between x and the harmonic sinusoid s_k
  - Therefore, $X[k]$ measures the "frequency content" of x at frequency k

$$X_u[k] \;=\; \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

- **Synthesis (Inverse DFT)**
  - Build up the signal x as a linear combination of harmonic sinusoids s_k weighted by the DFT coefficients $X[k]$

$$x[n] \;=\; \frac{1}{N}\sum_{k=0}^{N-1} X_u[k]\, e^{j\frac{2\pi}{N}kn}$$

The Centered DFT

- Both $x[n]$ and $X[k]$ are **periodic** with period N, so we can shift the intervals of interest in time and frequency to be centered around $n, k = 0$

$$-\frac{N}{2} \;\leq\; n, k \;\leq\; \frac{N}{2} - 1$$

- The modified forward and inverse DFT formulas are

$$X_u[k] \;=\; \sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad -\frac{N}{2} \;\leq\; k \;\leq\; \frac{N}{2} - 1$$

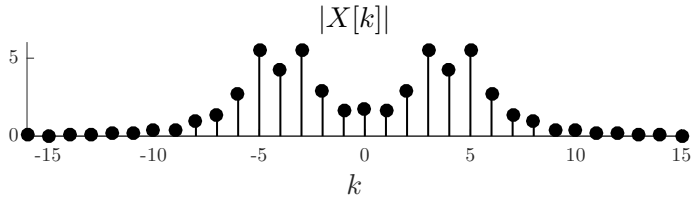
$$x[n] \;=\; \frac{1}{N} \sum_{k=-N/2}^{N/2-1} X_u[k]\, e^{j\frac{2\pi}{N}kn} \qquad -\frac{N}{2} \;\leq\; n \;\leq\; \frac{N}{2} - 1$$

Frequencies of the Centered DFT

$$X_u[k] \;=\; \sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad -\frac{N}{2} \;\leq\; k \;\leq\; \frac{N}{2}-1$$

- $X_u[k]$ measures the similarity between the time signal x and the harmonic sinusoid s_k
- Therefore, $X_u[k]$ measures the "frequency content" of x at frequency

$$-\pi \;\leq\; \omega_k = \frac{2\pi}{N}k \;<\; \pi$$

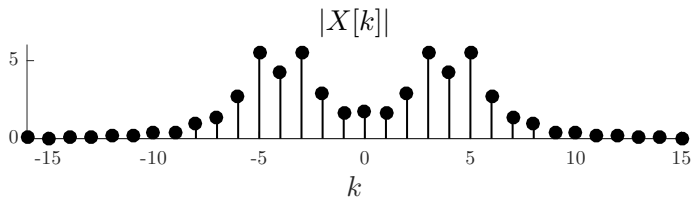


Take It To The Limit (1)

$$X_u[k] \;=\; \sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad -\frac{N}{2} \;\leq\; k \;\leq\; \frac{N}{2}-1$$

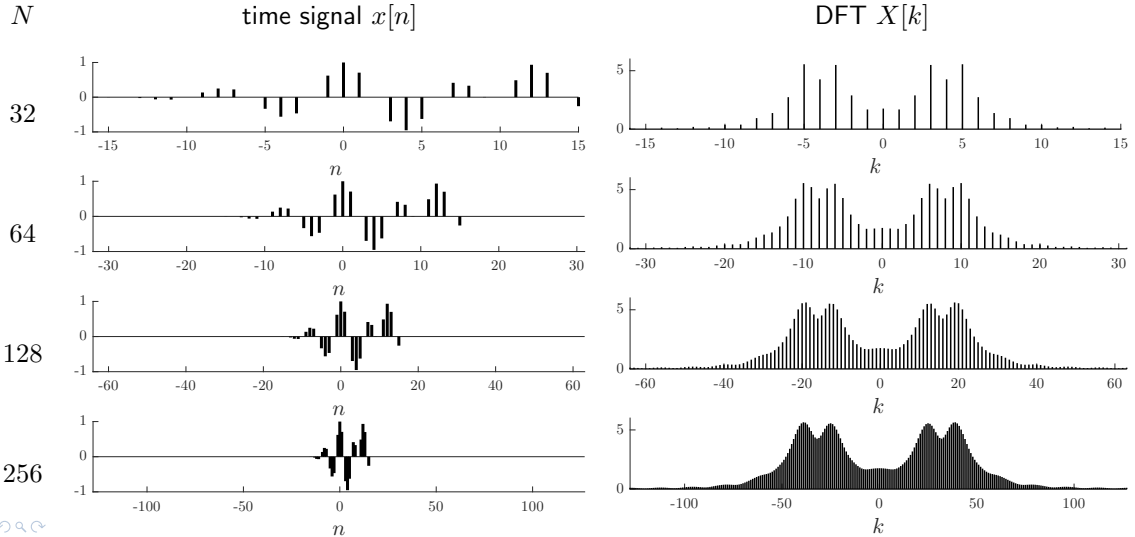
- Let the signal length N increase towards ∞ and study what happens to $X_u[k]$
- **Key fact:** No matter how large N grows, the frequencies of the DFT sinusoids remain in the interval

$$-\pi \;\leq\; \omega_k = \frac{2\pi}{N}k \;<\; \pi$$



Take It To The Limit (2)

$$X_u[k] = \sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$



Discrete Time Fourier Transform (Forward)

- As $N \to \infty$, the forward DFT converges to a function of the **continuous frequency variable** ω that we will call the **forward discrete time Fourier transform** (DTFT)

$$\sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \longrightarrow \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \;=\; X(\omega), \qquad -\pi \le \omega < \pi$$

- Recall: Inner product for infinite-length signals

$$\langle x, y \rangle \;=\; \sum_{n=-\infty}^{\infty} x[n]\, y[n]^*$$

- **Analysis interpretation:** The value of the DTFT $X(\omega)$ at frequency ω measures the similarity of the infinite-length signal $x[n]$ to the infinite-length sinusoid $e^{j\omega n}$

Discrete Time Fourier Transform (Inverse)

- Inverse unnormalized DFT

$$x[n] \;=\; \frac{2\pi}{2\pi N} \sum_{k=-N/2}^{N/2-1} X_u[k]\, e^{j\frac{2\pi}{N}kn}$$

- In the limit as the signal length $N \to \infty$, the inverse DFT converges in a more subtle way:

$$e^{j\frac{2\pi}{N}kn} \longrightarrow e^{j\omega n}, \qquad X_u[k] \longrightarrow X(\omega), \qquad \frac{2\pi}{N} \longrightarrow d\omega, \qquad \sum_{k=-N/2}^{N/2-1} \longrightarrow \int_{-\pi}^{\pi}$$

  resulting in the **inverse DTFT**

$$x[n] \;=\; \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}, \qquad \infty < n < \infty$$

- **Synthesis interpretation:** Build up the signal x as an infinite linear combination of sinusoids $e^{j\omega n}$ weighted by the DTFT $X(\omega)$

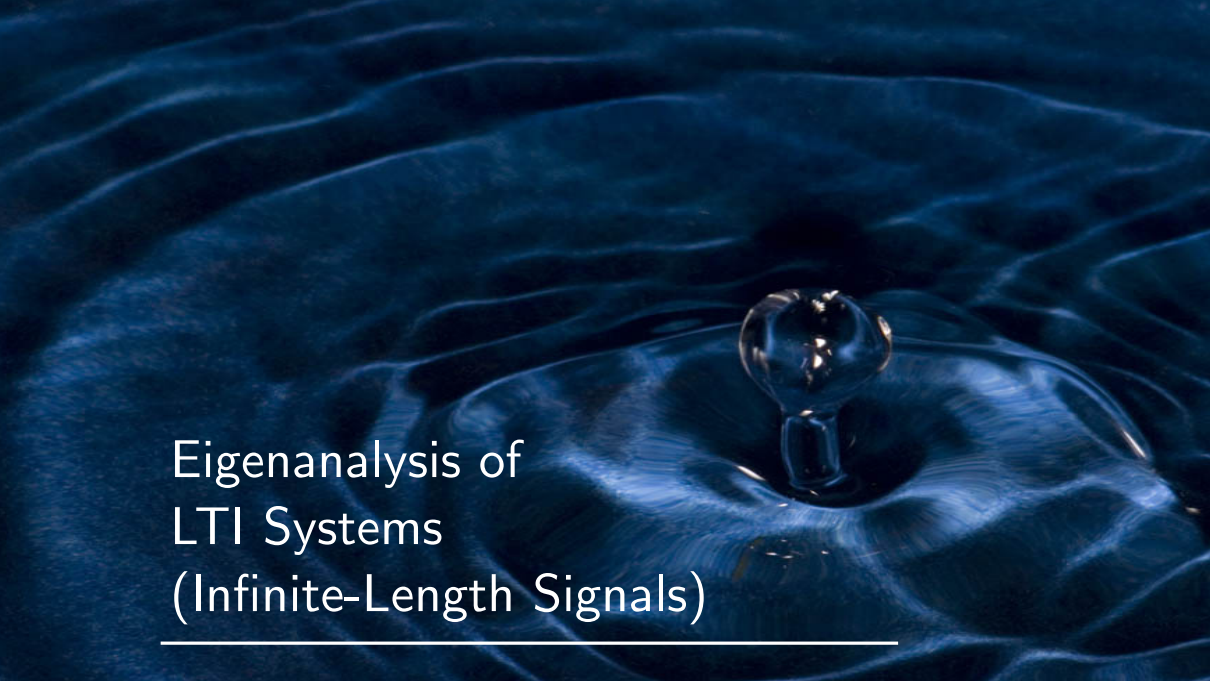
Summary

- Discrete-time Fourier transform (DTFT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad -\pi \leq \omega < \pi$$

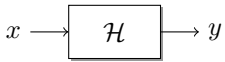
$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}, \qquad \infty < n < \infty$$

- The core "basis functions" of the DTFT are the sinusoids $e^{j\omega n}$ with arbitrary frequencies ω
- The DTFT can be derived as the limit of the DFT as the signal length $N \to \infty$
- The analysis/synthesis interpretation of the DFT holds for the DTFT, as do most of its properties



Eigenanalysis of LTI Systems (Infinite-Length Signals)

LTI Systems for Infinite-Length Signals



$$y \;=\; \mathbf{H}x$$

- For infinite length signals, $\mathbf{H}$ is an infinitely large **Toeplitz matrix** with entries

$$[\mathbf{H}]_{n,m} \;=\; h[n-m]$$

  where h is the **impulse response**

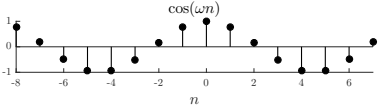
- **Goal:** Calculate the **eigenvectors** and **eigenvalues** of $\mathbf{H}$

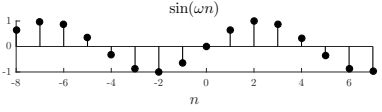
- Eigenvectors v are input signals that emerge at the system output unchanged (except for a scaling by the eigenvalue λ) and so are somehow "fundamental" to the system

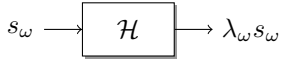
Eigenvectors of LTI Systems

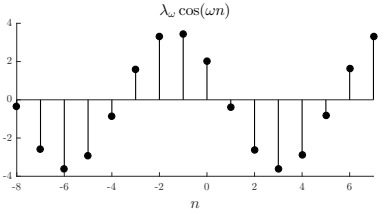
- **Fact:** The eigenvectors of a Toeplitz matrix (LTI system) are the complex **sinusoids**

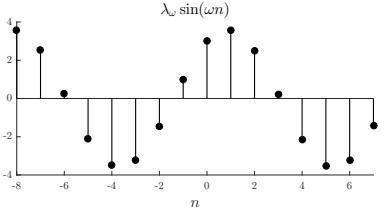
$$s_\omega[n] \;=\; e^{j\omega n} \;=\; \cos(\omega n) + j\sin(\omega n), \qquad -\pi \le \omega < \pi, \quad -\infty < n < \infty$$



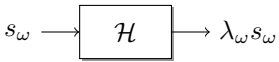








Sinusoids are Eigenvectors of LTI Systems



- Prove that harmonic sinusoids are the eigenvectors of LTI systems simply by computing the convolution with input s_ω and applying the periodicity of the sinusoids (infinite-length)

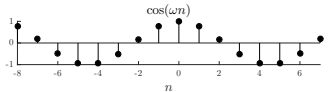
$$
\begin{aligned}
s_\omega[n] * h[n] &= \sum_{m=-\infty}^{\infty} s_\omega[n-m]\, h[m] = \sum_{m=-\infty}^{\infty} e^{j\omega(n-m)}\, h[m] \\
&= \sum_{m=-\infty}^{\infty} e^{j\omega n}\, e^{-j\omega m}\, h[m] = \left(\sum_{m=-\infty}^{\infty} h[m]\, e^{-j\omega m} \right) e^{j\omega n} \\
&= \lambda_\omega\, s_\omega[n] \quad \checkmark
\end{aligned}
$$

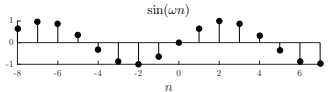
Eigenvalues of LTI Systems

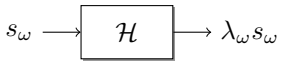
- The eigenvalue $\lambda_\omega \in \mathbb{C}$ corresponding to the sinusoid eigenvector s_ω is called the **frequency response** at frequency ω since it measures how the system "responds" to s_k

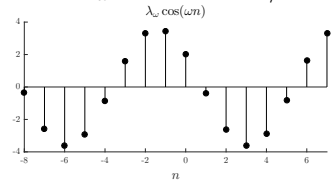
$$\lambda_\omega \;=\; \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} \;=\; \langle h, s_\omega \rangle \;=\; H(\omega) \;\; \text{(DTFT of } h\text{)}$$

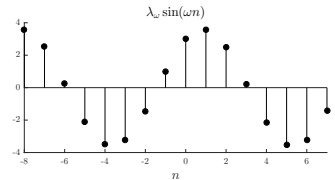
- Recall properties of the **inner product**: λ_ω grows/shrinks as h and s_ω become more/less similar



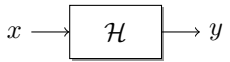








Eigendecomposition and Diagonalization of an LTI System



$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- While we can't explicitly display the infinitely large matrices involved, we can use the DTFT to "diagonalize" an LTI system
- Taking the DTFTs of x and h

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad H(\omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

we have that

$$Y(\omega) \;=\; X(\omega) H(\omega)$$

and then

$$y[n] = \int_{-\pi}^{\pi} Y(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

Summary

- Complex sinusoids are the eigenfunctions of LTI systems for infinite-length signals (Toeplitz matrices)
- Therefore, the discrete time Fourier transform (DTFT) is the natural tool for studying LTI systems for infinite-length signals
- Frequency response $H(\omega)$ equals the DTFT of the impulse response $h[n]$
- Diagonalization by eigendecomposition implies

$$Y(\omega) \;=\; X(\omega)\, H(\omega)$$

Discrete Time Fourier Transform Examples

Discrete Time Fourier Transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad -\pi \leq \omega < \pi$$

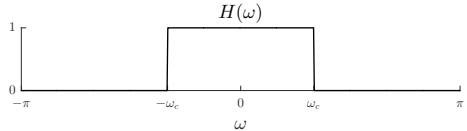
$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}, \qquad \infty < n < \infty$$

- The Fourier transform of choice for analyzing infinite-length signals and systems
- Useful for conceptual, pencil-and-paper work, but not Matlab friendly (infinitely long vectors)

Impulse Response of the Ideal Lowpass Filter (1)

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$



- Compute the impulse response $h[n]$ given this $H(\omega)$

- Apply the inverse DTFT

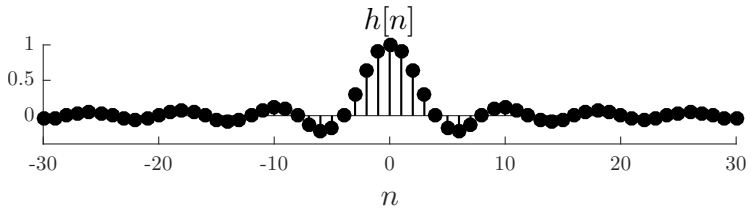
$$h[n] = \int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n}\, \frac{d\omega}{2\pi} = \left. \frac{e^{j\omega n}}{2\pi j n} \right|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

Impulse Response of the Ideal Lowpass Filter (2)

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

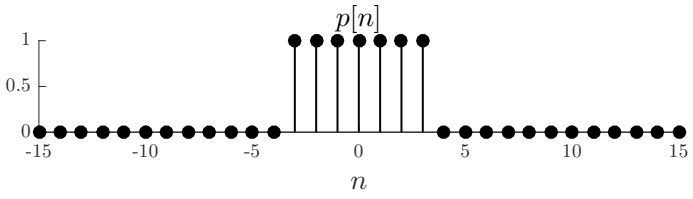
$$h[n] \;=\; 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



- The infamous "sinc" function!

DTFT of a Moving Average System

- Compute the DTFT of the symmetrical **moving average system** $p[n] = \begin{cases} 1 & -M \le n \le M \\ 0 & \text{otherwise} \end{cases}$
- Note: Duration $D_x = 2M + 1$ samples
- Example for $M = 3$



- Forward DTFT

$$P(\omega) = \sum_{n=-\infty}^{\infty} p[n]\, e^{-j\omega n} = \sum_{n=-M}^{M} e^{-j\omega n} \quad \ldots$$

DTFT of the Unit Pulse (2)

- Apply the finite geometric series formula

$$P(\omega) = \sum_{n=-\infty}^{\infty} p[n]\, e^{-j\omega n} = \sum_{n=-M}^{M} e^{-j\omega n} = \sum_{n=-M}^{M} \left(e^{-j\omega}\right)^n = \frac{e^{j\omega M} - e^{-j\omega(M+1)}}{1 - e^{-j\omega}}$$

- This is an answer but it is not simplified enough to make sense, so we continue simplifying

$$\begin{aligned} P(\omega) &= \frac{e^{j\omega M} - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} = \frac{e^{-j\omega/2}\left(e^{j\omega\frac{2M+1}{2}} - e^{-j\omega\frac{2M+1}{2}}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} \\ &= \frac{2j\sin\left(\omega\frac{2M+1}{2}\right)}{2j\sin\left(\frac{\omega}{2}\right)} \end{aligned}$$

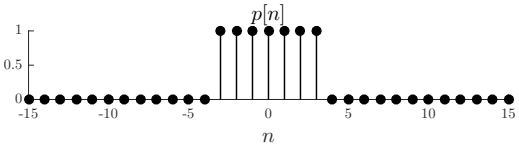
DTFT of the Unit Pulse (3)

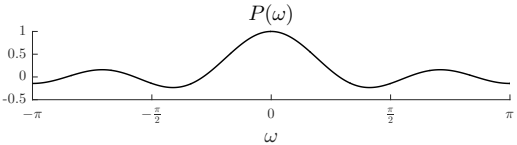
- Simplified DTFT of the unit pulse of duration $D_x = 2M + 1$ samples

$$P(\omega) \;=\; \frac{\sin\left(\frac{2M+1}{2}\,\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

- This is called the **Dirichlet kernel** or "digital sinc"
  - It has a shape reminiscent of the classical $\sin x/x$ sinc function, but it is 2π-periodic

- If $p[n]$ is interpreted as the impulse response of the moving average system, then $P(\omega)$ is the frequency response (eigenvalues) (low-pass filter)



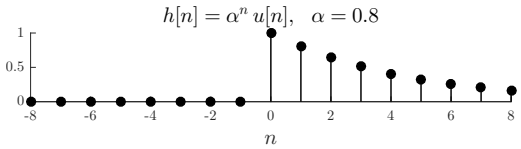


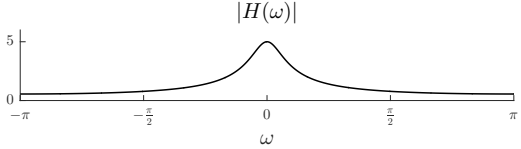
DTFT of a One-Sided Exponential

- Recall the impulse response of the recursive average system: $h[n] = \alpha^n\, u[n],\ |\alpha| < 1$
- Compute the frequency response $H(\omega)$
- Forward DTFT

$$H(\omega) \;=\; \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} \;=\; \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n} \;=\; \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n \;=\; \frac{1}{1-\alpha\, e^{-j\omega}}$$

- Recursive system with $\alpha = 0.8$ is a low-pass filter





Summary

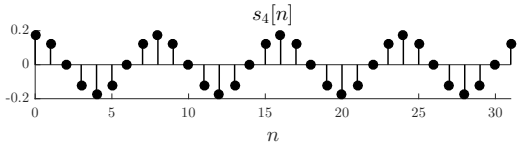
- DTFT of a rectangular pulse is a Dirichlet kernel
- DTFT of a one-sided exponential is a low-frequency bump
- Inverse DTFT of the ideal lowpass filter is a sinc function
- Work some examples on your own!

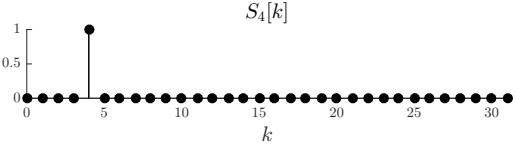
Discrete Time Fourier Transform of a Sinusoid

Discrete Fourier Transform (DFT) of a Harmonic Sinusoid

- Thanks to the orthogonality of the length-N harmonic sinusoids, it is easy to calculate the DFT of the harmonic sinusoid $x[n] = s_l[n] = e^{j\frac{2\pi}{N}ln}/\sqrt{N}$

$$X[k] \;=\; \sum_{n=0}^{N-1} s_l[n]\,\frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}} \;=\; \langle s_l, s_k\rangle \;=\; \delta[k-l]$$





- So what is the DTFT of the infinite length sinusoid $e^{j\omega_0 n}$?

DTFT of an Infinite-Length Sinusoid

- The calculation for the DTFT and infinite-length signals is much more delicate than for the DFT and finite-length signals

- Calculate the value $X(\omega_0)$ for the signal $x[n] = e^{j\omega_0 n}$

$$X(\omega_0) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega_0 n} \;=\; \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}\, e^{-j\omega_0 n} \;=\; \sum_{n=-\infty}^{\infty} 1 \;=\; \infty$$

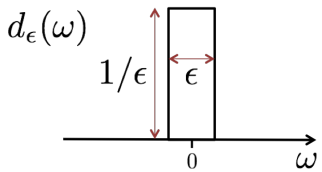
- Calculate the value $X(\omega)$ for the signal $x[n] = e^{j\omega_0 n}$ at a frequency $\omega \neq \omega_0$

$$X(\omega_0) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \;=\; \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}\, e^{-j\omega n} \;=\; \sum_{n=-\infty}^{\infty} e^{-j(\omega-\omega_0)n} \;=\; ???$$

Dirac Delta Function (1)

- One semi-rigorous way to deal with this quandary is to use the **Dirac delta "function,"** which is defined in terms of the following limit process

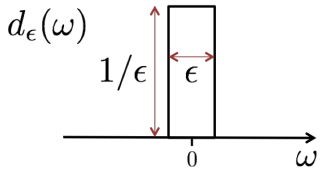
- Consider the following function $d_\epsilon(\omega)$ of the continuous variable ω



- Note that, for all values of the width ϵ, $d_\epsilon(\omega)$ always has unit area

$$\int d_\epsilon(\omega)\, d\omega \;=\; 1$$

Dirac Delta Function (2)

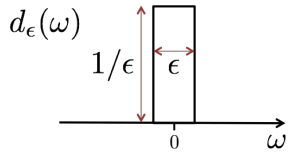


- What happens to $d_\epsilon(\omega)$ as we let $\epsilon \to 0$?
  - Clearly $d_\epsilon(\omega)$ is converging toward something that is infinitely tall and infinitely narrow but still with unit area

- The safest way to handle a function like $d_\epsilon(\omega)$ is inside an integral, like so

$$\int X(\omega)\, d_\epsilon(\omega)\, d\omega$$

Dirac Delta Function (3)



- As $\epsilon \to 0$, it seems reasonable that

$$\int X(\omega)\, d_\epsilon(\omega)\, d\omega \overset{\epsilon \to 0}{\longrightarrow} X(0)$$

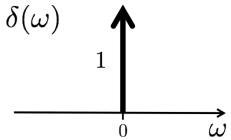
and

$$\int X(\omega)\, d_\epsilon(\omega - \omega_0)\, d\omega \overset{\epsilon \to 0}{\longrightarrow} X(\omega_0)$$

- So we can think of $d_\epsilon(\omega)$ as a kind of "sampler" that picks out values of functions from inside an integral
- We describe the results of this limiting process (as $\epsilon \to 0$) as the **Dirac delta "function"** $\delta(\omega)$

Dirac Delta Function (4)

- **Dirac delta "function"** $\delta(\omega)$



- We write

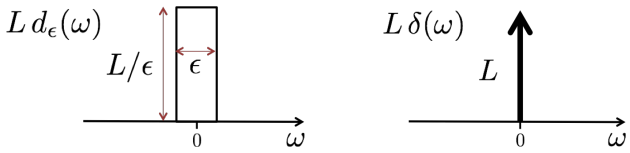
$$\int X(\omega)\,\delta(\omega)\,d\omega \;=\; X(0)$$

and

$$\int X(\omega)\,\delta(\omega-\omega_0)\,d\omega \;=\; X(\omega_0)$$

- Remarks and caveats
  - Do not confuse the Dirac delta "function" with the nicely behaved discrete delta function $\delta[n]$
  - The Dirac has lots of "delta," but it is not really a "function" in the normal sense (it can be made more rigorous using the theory of generalized functions)
  - Colloquially, engineers will describe the Dirac delta as "infinitely tall and infinitely narrow"

Scaled Dirac Delta Function



- If we scale the area of $d_\epsilon(\omega)$ by L, then it has the following effect in the limit

$$\int X(\omega)\, L\, \delta(\omega)\, d\omega \;=\; L\, X(0)$$

And Now Back to Our Regularly Scheduled Program . . .

- Back to determining the DTFT of an infinite length sinusoid

- Rather than computing the DTFT of a sinusoid using the forward DTFT, we will show that an infinite-length sinusoid is the inverse DTFT of the scaled Dirac delta function $2\pi\delta(\omega - \omega_0)$

$$\int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_0)\, e^{j\omega n}\, \frac{d\omega}{2\pi} \;=\; e^{j\omega_0 n}$$

- Thus we have the (rather bizarre) DTFT pair

$$e^{j\omega_0 n} \;\overset{\text{DTFT}}{\longleftrightarrow}\; 2\pi\,\delta(\omega - \omega_0)$$

DTFT of Real-Valued Sinusoids

- Since

$$\cos(\omega_0 n) \;=\; \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)$$

we can calculate its DTFT as

$$\cos(\omega_0 n) \;\overset{\text{DTFT}}{\longleftrightarrow}\; \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$

- Since

$$\sin(\omega_0 n) \;=\; \frac{1}{2j}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right)$$

we can calculate its DTFT as

$$\sin(\omega_0 n) \;\overset{\text{DTFT}}{\longleftrightarrow}\; \frac{\pi}{j}\,\delta(\omega-\omega_0) + \frac{\pi}{j}\,\delta(\omega+\omega_0)$$

Summary

- The DTFT would be of limited utility if we could not compute the transform of an infinite-length sinusoid
- Hence, the Dirac delta "function" (or something else) is a necessary evil
- The Dirac delta has infinite energy (2-norm); but then again so does an infinite-length sinusoid

Discrete Time Fourier Transform Properties

Recall: Discrete-Time Fourier Transform (DTFT)

- Forward DTFT (Analysis)

$$X(\omega) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad -\pi \le \omega < \pi$$

- Inverse DTFT (Synthesis)

$$x[n] \;=\; \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}, \qquad \infty < n < \infty$$

- DTFT pair

$$x[n] \;\overset{\text{DTFT}}{\longleftrightarrow}\; X(\omega)$$

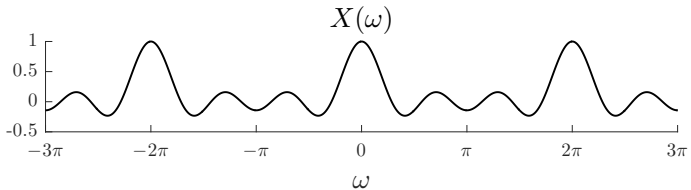
The DTFT is Periodic

- We defined the DTFT over an interval of ω of length 2π, but it can also be interpreted as **periodic** with period 2π

$$X(\omega) \;=\; X(\omega + 2\pi k), \quad k \in \mathbb{Z}$$

- Proof

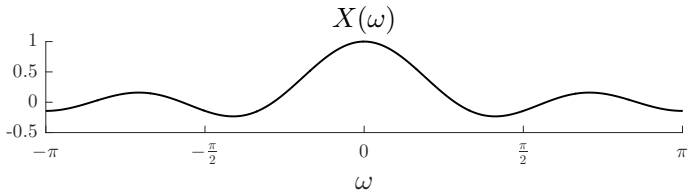
$$X(\omega + 2\pi k) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi k)n} \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\, e^{-j2\pi kn} \;=\; X(\omega) \quad \checkmark$$



DTFT Frequencies

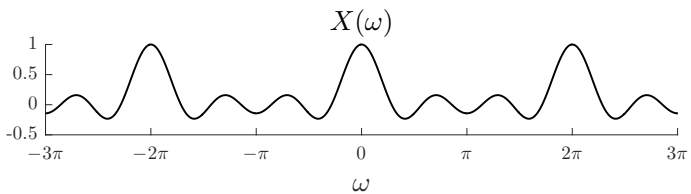
$$X(\omega) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad -\pi \le \omega < \pi$$

- $X(\omega)$ measures the similarity between the time signal x and and a sinusoid $e^{j\omega n}$ of frequency ω
- Therefore, $X(\omega)$ measures the "frequency content" of x at frequency ω



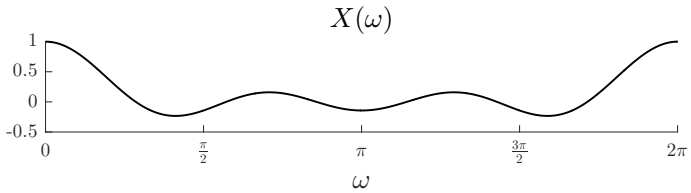
DTFT Frequencies and Periodicity

- Periodicity of DTFT means we can treat frequencies mod 2π
- $X(\omega)$ measures the "frequency content" of x at frequency $(\omega)_{2\pi}$

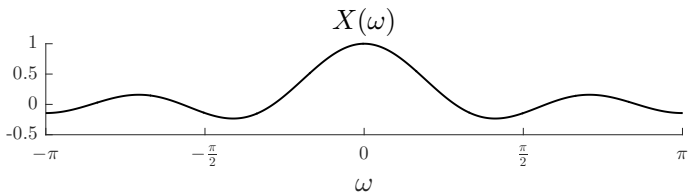


DTFT Frequency Ranges

- Periodicity of DTFT means every length-2π interval of ω carries the same information
- Typical interval 1: $0 \leq \omega < 2\pi$



- Typical interval 2: $-\pi \leq \omega < \pi$ (more intuitive)



DTFT and Time Shift

- If $x[n]$ and $X(\omega)$ are a DTFT pair then

$$x[n-m] \overset{\text{DTFT}}{\longleftrightarrow} e^{-j\omega m} X(\omega)$$

- Proof: Use the change of variables $r = n - m$

$$\sum_{n=-\infty}^{\infty} x[n-m]\, e^{-j\omega n} = \sum_{r=-\infty}^{\infty} x[r]\, e^{-j\omega(r+m)} = \sum_{r=-\infty}^{\infty} x[r]\, e^{-j\omega r}\, e^{-j\omega m}$$

$$= e^{-j\omega m} \sum_{r=-\infty}^{\infty} x[r]\, e^{-j\omega r} = e^{-j\omega m}\, X(\omega) \quad \checkmark$$

DTFT and Modulation

- If $x[n]$ and $X(\omega)$ are a DTFT pair then

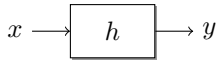
$$e^{j\omega_0 n}\, x[n] \overset{\text{DTFT}}{\longleftrightarrow} X(\omega - \omega_0)$$

- Remember that the DTFT is 2π-periodic, and so we can interpret the right hand side as $X((\omega - \omega_0)_{2\pi})$

- Proof:

$$\sum_{n=-\infty}^{\infty} e^{j\omega_0 n}\, x[n]\, e^{-j\omega n} \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega-\omega_0)n} \;=\; X(\omega - \omega_0) \quad \checkmark$$

DTFT and Convolution



$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- If

$$x[n] \overset{\text{DTFT}}{\longleftrightarrow} X(\omega), \qquad h[n] \overset{\text{DTFT}}{\longleftrightarrow} H(\omega), \qquad y[n] \overset{\text{DTFT}}{\longleftrightarrow} Y(\omega)$$

  then

$$Y(\omega) \;=\; H(\omega)\, X(\omega)$$

- Convolution in the time domain $\Leftrightarrow$ multiplication in the frequency domain

The DTFT is Linear

- It is trivial to show that if

$$x_1[n] \overset{\text{DTFT}}{\longleftrightarrow} X_1(\omega) \qquad x_2[n] \overset{\text{DTFT}}{\longleftrightarrow} X_2(\omega)$$

then

$$\alpha_1 x_1[n] + \alpha_2 x[2] \overset{\text{DTFT}}{\longleftrightarrow} \alpha_1 X_1(\omega) + \alpha_2 X_2(\omega)$$

DTFT Symmetry Properties

- The sinusoids $e^{j\omega n}$ of the DTFT have symmetry properties:

$$\begin{aligned} \mathrm{Re}\left(e^{j\omega n}\right) &= \cos(\omega n) \quad \text{(even function)} \\ \mathrm{Im}\left(e^{j\omega n}\right) &= \sin(\omega n) \quad \text{(odd function)} \end{aligned}$$

- These induce corresponding symmetry properties on $X(\omega)$ around the frequency $\omega = 0$

- **Even** signal/DFT

$$x[n] = x[-n], \qquad X(\omega) = X(-\omega)$$

- **Odd** signal/DFT

$$x[n] = -x[-n], \qquad X(\omega) = -X(-\omega)$$

- Proofs of the symmetry properties are identical to the DFT case; omitted here

DTFT Symmetry Properties Table

$x[n]$	$X(\omega)$	$\mathrm{Re}(X(\omega))$	$\mathrm{Im}(X(\omega))$	$\lvert X(\omega)\rvert$	$\angle X(\omega)$
real	$X(-\omega) = X(\omega)^*$	even	odd	even	odd
real & even	real & even	even	zero	even	
real & odd	imaginary & odd	zero	odd	even	
imaginary	$X(-\omega) = -X(\omega)^*$	odd	even	even	odd
imaginary & even	imaginary & even	zero	even	even	
imaginary & odd	real & odd	odd	zero	even	

Summary

- DTFT is periodic with period 2π
- Convolution in time becomes multiplication in frequency
- DTFT has useful symmetry properties

z-Transform

Set G

z-Transform

z-Transform

- The z-transform generalizes the Discrete-Time Fourier Transform (DTFT) for analyzing infinite-length signals and systems
- Very useful for designing and analyzing signal processing systems
- Properties are very similar to the DTFT with a few caveats
- The theme this week is less linear algebra (vectors spaces) and more polynomial algebra

Recall: DTFT

- Discrete-time Fourier transform (DTFT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad -\pi \leq \omega < \pi$$

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}, \qquad \infty < n < \infty$$

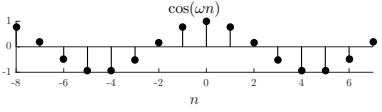
- The core "basis functions" of the DTFT are the sinusoids $e^{j\omega n}$ with arbitrary frequencies ω

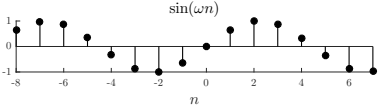
- The sinusoids $e^{j\omega n}$ are eigenvectors of LTI systems for infinite-length signals (infinite Toeplitz matrices)

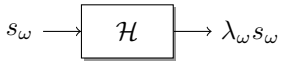
Recall: Complex Sinusoids are Eigenvectors of LTI Systems

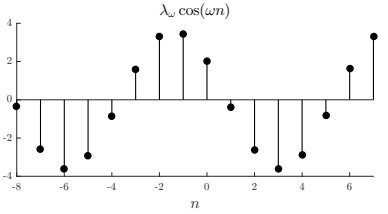
- **Fact:** The eigenvectors of a Toeplitz matrix (LTI system) are the complex **sinusoids**

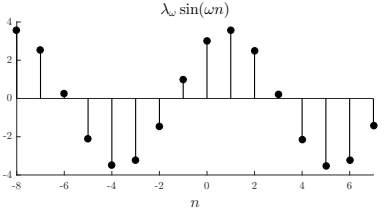
$$s_\omega[n] \;=\; e^{j\omega n} \;=\; \cos(\omega n) + j\sin(\omega n), \qquad -\pi \le \omega < \pi, \quad -\infty < n < \infty$$









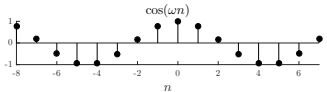


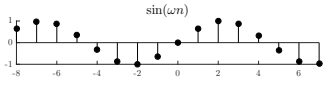
Recall: Eigenvalues of LTI Systems

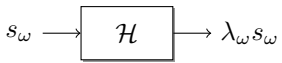
- The eigenvalue $\lambda_\omega \in \mathbb{C}$ corresponding to the sinusoid eigenvector s_ω is called the **frequency response** at frequency ω since it measures how the system "responds" to s_k

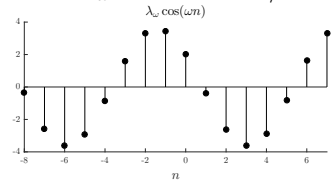
$$\lambda_\omega \;=\; \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} \;=\; \langle h, s_\omega \rangle \;=\; H(\omega) \;\; \text{(DTFT of } h\text{)}$$

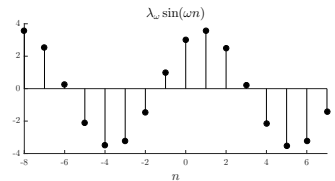
- Recall properties of the **inner product**: λ_ω grows/shrinks as h and s_ω become more/less similar



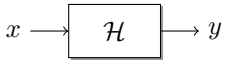








Eigendecomposition and Diagonalization of an LTI System



$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- While we can't explicitly display the infinitely large matrices involved, we can use the DTFT to "diagonalize" an LTI system
- Taking the DTFTs of x and h

$$X(\omega) = \sum_{m=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad H(\omega) = \sum_{m=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

we have that

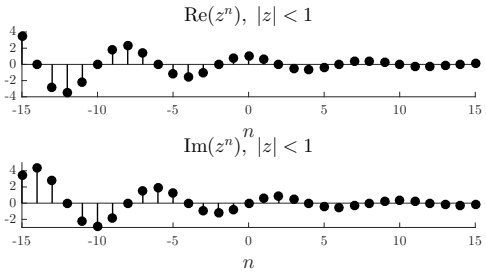
$$Y(\omega) \;=\; X(\omega)H(\omega)$$

Recall: Complex Exponential

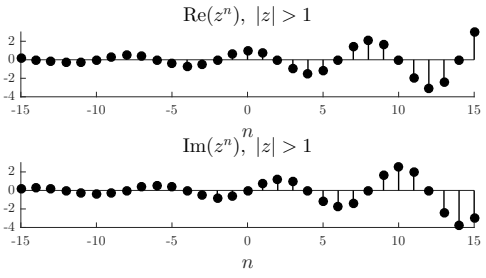
$$z^n \;=\; \left(|z|\, e^{j\omega}\right)^n \;=\; |z|^n\, e^{j\omega n} \;=\; |z|^n\, \left(\cos(\omega n) + j\sin(\omega n)\right)$$

- $|z|^n$ is a **real exponential** envelope $\left(a^n \text{ with } a = |z|\right)$
- $e^{j\omega n}$ is a **complex sinusoid**

$|z| < 1$

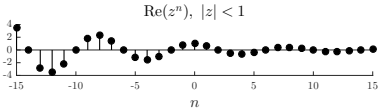


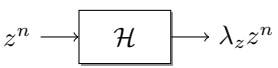
$|z| > 1$

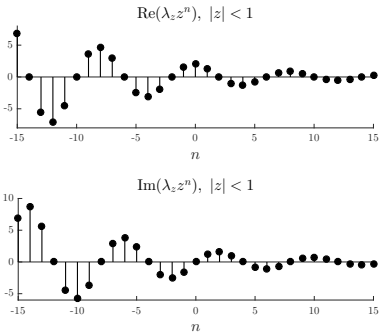


Complex Exponentials are Eigenvectors of LTI Systems

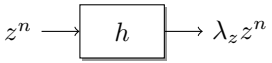
- **Fact:** A more general set of eigenvectors of a Toeplitz matrix (LTI system) are the **complex exponentials** z^n, $z \in \mathbb{C}$







Proof: Complex Exponentials are Eigenvectors of LTI Systems



- Prove that complex exponentials are the eigenvectors of LTI systems simply by computing the convolution with input z^n

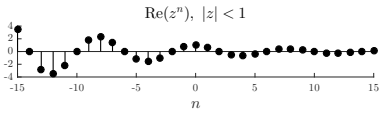
$$
\begin{aligned}
z^n * h[n] &= \sum_{m=-\infty}^{\infty} z^{n-m}\, h[m] = \sum_{m=-\infty}^{\infty} z^n\, z^{-m}\, h[m] \\
&= \left(\sum_{m=-\infty}^{\infty} h[m]\, z^{-m} \right) z^n \\
&= \lambda_z\, z^n \quad \checkmark
\end{aligned}
$$

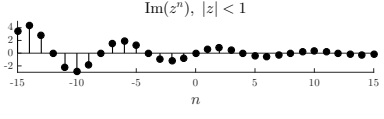
Eigenvalues of LTI Systems

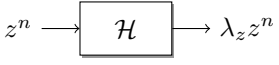
- The eigenvalue $\lambda_z \in \mathbb{C}$ corresponding to the complex exponential eigenvector z^n is called the **transfer function**; it measures how the system "transfers" the input z^n to the output

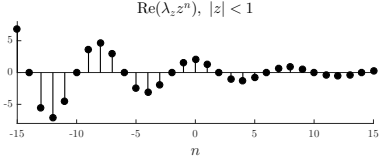
$$\lambda_z \;=\; \sum_{n=-\infty}^{\infty} h[n]\, z^{-n} \;=\; H(z)$$

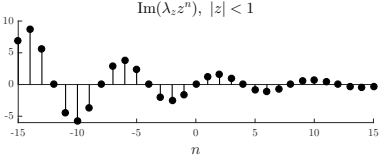
- Recall properties of the **inner product**: λ_z grows/shrinks as $h[n]$ becomes more/less similar to $(z^{-n})^*$











z-Transform

- Define the (bilateral) **forward z-transform** of $x[n]$ as

$$X(z) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- The core "basis functions" of the z-transform are the complex exponentials z^n with arbitrary $z \in \mathbb{C}$; these are the eigenvectors of LTI systems for infinite-length signals

- **Notation abuse alert:** We use $X(\cdot)$ to represent both the DTFT $X(\omega)$ and the z-transform $X(z)$; they are, in fact, intimately related

$$X_{\mathrm{DTFT}}(\omega) \;=\; X_z(z)|_{z=e^{j\omega}} \;=\; X_z(e^{j\omega})$$

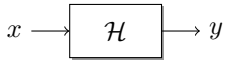
(Note how this elegantly produces a 2π-periodic DTFT)

z-Transform as a Function

$$X(z) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- $X(z)$ is a complex-valued function of a complex variable: $\quad X(z) \in \mathbb{C}, \quad z \in \mathbb{C}$

Eigendecomposition and Diagonalization of an LTI System



$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

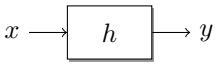
- While we can't explicitly display the infinitely large matrices involved, we can use the z-transform to "diagonalize" an LTI system
- Taking the z-transforms of x and h

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}, \qquad H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

we have that

$$Y(z) \;=\; X(z)\, H(z)$$

Proof: Eigendecomposition and Diagonalization of an LTI System



- Compute the z-transform of the result of the convolution of x and h
  (Note: we are a little cavalier about exchanging the infinite sums below)

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x[m]\, h[n-m] \right) z^{-n} \\
&= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{n=-\infty}^{\infty} h[n-m]\, z^{-n} \right) \quad (\text{let } r = n-m) \\
&= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{r=-\infty}^{\infty} h[r]\, z^{-r-m} \right) = \left(\sum_{m=-\infty}^{\infty} x[m] z^{-m} \right) \left(\sum_{r=-\infty}^{\infty} h[r]\, z^{-r} \right) \\
&= X(z)\, H(z) \quad \checkmark
\end{aligned}
$$

Summary

- Complex exponentials z^n are the eigenfunctions of LTI systems for infinite-length signals (Toeplitz matrices)
- Forward z-transform

$$X(z) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- Transfer function $H(z)$ equals the z-transform of the impulse response $h[n]$
- Diagonalization by eigendecomposition implies

$$Y(z) \;=\; X(z)\, H(z)$$

- DTFT is a special case of the z-transform (values on the unit circle in the complex z-plane)

$$X_{\mathrm{DTFT}}(\omega) \;=\; X_z(z)\big|_{z=e^{j\omega}} \;=\; X_z(e^{j\omega})$$

(Note how this elegantly produces a 2π-periodic DTFT)

z-Transform
Region of Convergence

z-Transform

- The **forward z-transform** of the signal $x[n]$

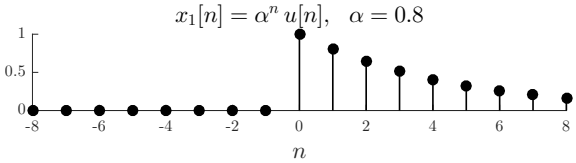
$$X(z) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- We have been a bit cavalier in our development regarding the convergence of the infinite sum that defines $X(z)$

- Work out two examples to expose the issues

Example 1: z-Transform of $\alpha^n u[n]$

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal)
- Example for $\alpha = 0.8$



- The **forward z-transform** of $x_1[n]$

$$X_1(z) \;=\; \sum_{n=-\infty}^{\infty} x_1[n]\, z^{-n} \;=\; \sum_{n=0}^{\infty} \alpha^n\, z^{-n} \;=\; \sum_{n=0}^{\infty} (\alpha\, z^{-1})^n \;=\; \frac{1}{1-\alpha\, z^{-1}} \;=\; \frac{z}{z-\alpha}$$

- **Important:** We can apply the geometric sum formula only when $|\alpha\, z^{-1}| < 1$ or $|z| > |\alpha|$

Region of Convergence

DEFINITION

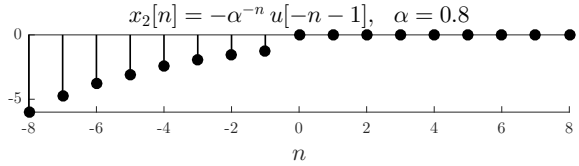
Given a time signal $x[n]$, the **region of convergence** (ROC) of its z-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\, z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| \; < \; \infty$$

- Example: For $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$, the ROC of $X_1(z) = \frac{1}{1-\alpha\, z^{-1}}$ is all z such that $|z| > |\alpha|$

Example 2: z-Transform of $-\alpha^n u[-n-1]$

- Signal $x_2[n] = -\alpha^n u[-n-1]$, $\alpha \in \mathbb{C}$ (anti-causal signal)
- Example for $\alpha = 0.8$



- The **forward z-transform** of $x_2[n]$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n]\, z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n\, z^{-n} = -\sum_{n=1}^{\infty} \alpha^{-n}\, z^n = -\sum_{n=0}^{\infty} (\alpha^{-1}\, z)^n + 1$$

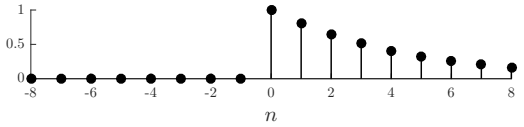
$$= \frac{-1}{1-\alpha^{-1}\, z} + 1 = \frac{-1}{1-\alpha^{-1}\, z} + \frac{1-\alpha^{-1}\, z}{1-\alpha^{-1}\, z} = \frac{-\alpha^{-1} z}{1-\alpha^{-1}\, z} = \frac{1}{1-\alpha\, z^{-1}} = \frac{z}{z-\alpha}$$

- ROC: $|\alpha^{-1}\, z| < 1$ or $|z| < |\alpha|$

Example Summary

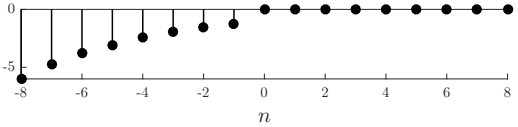
$$x_1[n] = \alpha^n u[n]$$

causal signal



$$x_2[n] = -\alpha^n u[-n-1]$$

anti-causal signal



$$X_1(z) \;=\; \frac{1}{1-\alpha\, z^{-1}} \;=\; X_2(z)$$

ROC

$|z| > |\alpha|$

ROC

$|z| < |\alpha|$

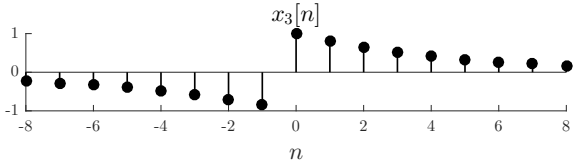
ROC: What We've Learned So Far

- The ROC is important
  - A z-transform converges only for certain values of z and does not exist for other values of z
  - z-transforms are non-unique without it

- You must always state the ROC when you state a z-transform

Example 3: z-Transform of $\alpha_1^n u[n] - \alpha_2^n u[-n-1]$

- Signal $x_3[n] = \alpha_1^n u[n] - \alpha_2^n u[-n-1]$, $\alpha_1, \alpha_2 \in \mathbb{C}$
- Example for $\alpha_1 = 0.8$ and $\alpha_2 = 1.2$



- The **forward z-transform** of the signal $x_3[n]$

$$X_3(z) = \sum_{n=-\infty}^{\infty} x_3[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} \alpha_1^n u[n]\, z^{-n} - \sum_{n=-\infty}^{\infty} \alpha_2^n u[-n-1]\, z^{-n}$$

$$= \frac{1}{1-\alpha_1\, z^{-1}} + \frac{1}{1-\alpha_2\, z^{-1}}$$

- ROC: $|z| > |\alpha_1|$ and $|z| < |\alpha_2|$

Example 3: A Tale of Two ROCs

- Signal $x_3[n] = \alpha_1^n u[n] - \alpha_2^n u[-n-1]$, $\alpha_1, \alpha_2 \in \mathbb{C}$
- z-transform

$$X_3(z) \;=\; \frac{1}{1-\alpha_1\, z^{-1}} \;+\; \frac{1}{1-\alpha_2\, z^{-1}}, \qquad |\alpha_1| < |z| < |\alpha_2|$$

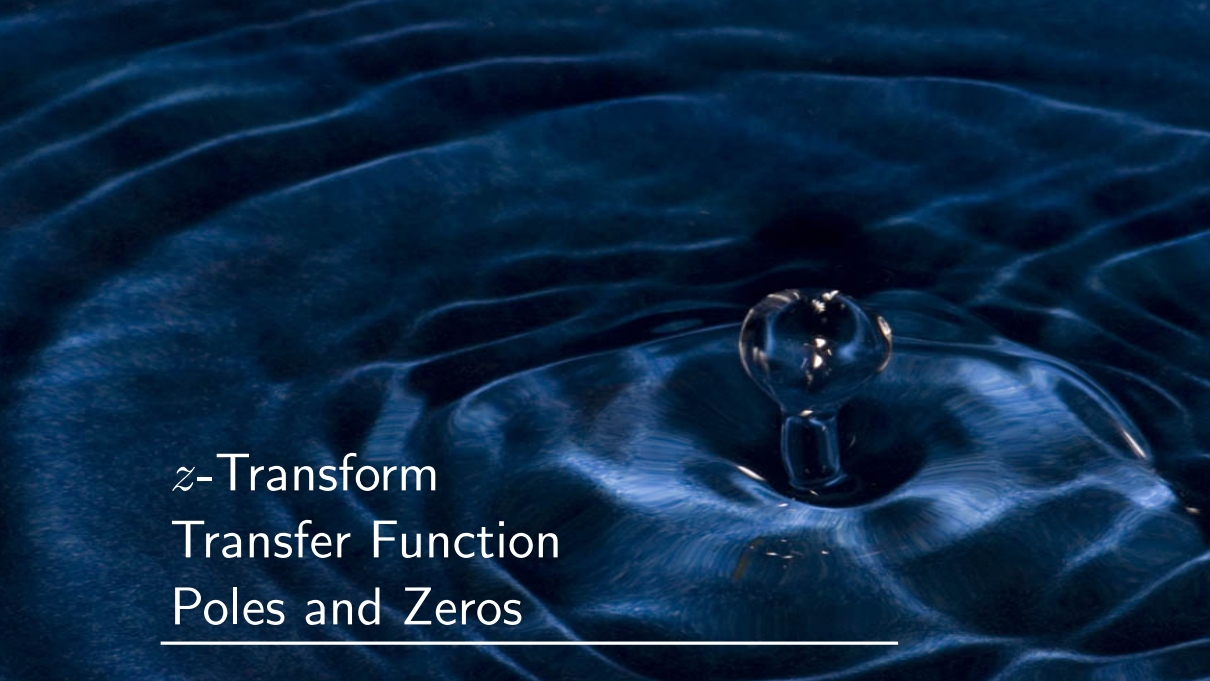
- Case 1: $|\alpha_1| < |\alpha_2|$
- Case 2: $|\alpha_1| > |\alpha_2|$

Properties of the ROC

- The ROC is a connected annulus (doughnut) in the z-plane centered on the origin $z = 0$; that is, the ROC is of the form $r_1 < |z| < r_2$
- If $x[n]$ has finite duration, then the ROC is the entire z-plane (except possibly $z = 0$ or $z = \infty$)
- If $x[n]$ is causal, then the ROC is the outside of a disk
- If $x[n]$ is anti-causal, then the ROC is the inside of a disk
- If $x[n]$ is two-sided (neither causal nor anti-causal), then either the ROC is the inside of an annulus or the z-transform does not converge
- The ROC is a connected region of the z-plane

Summary

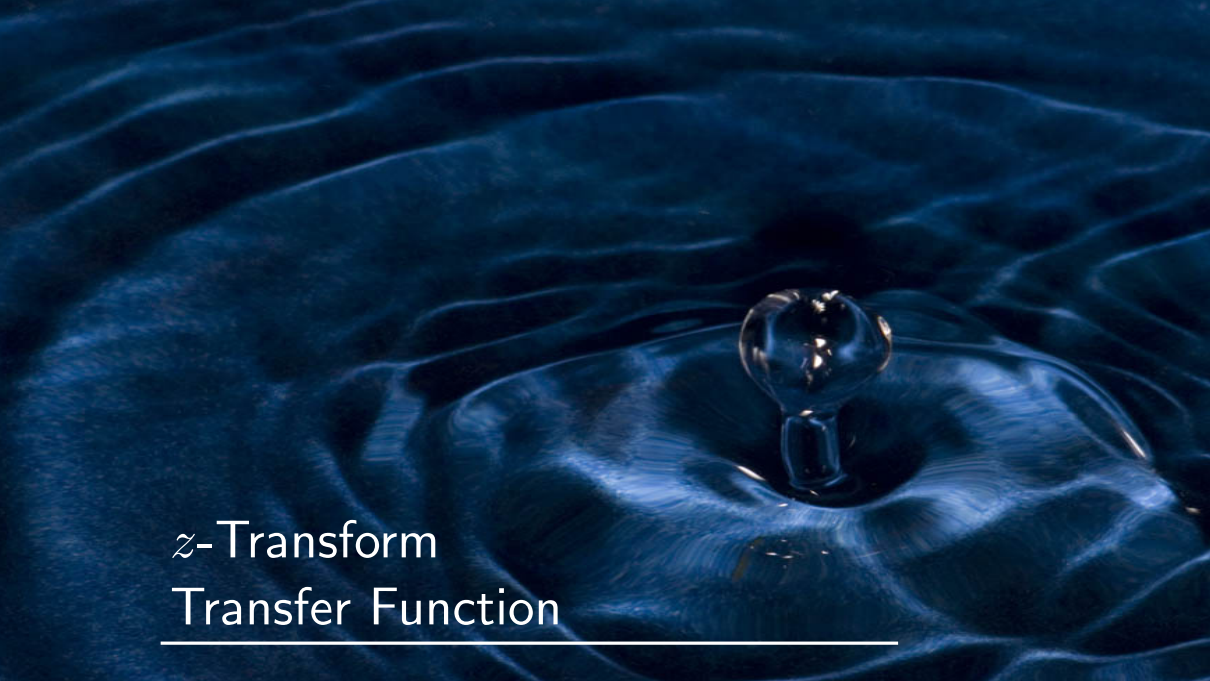
- The **region of convergence** (ROC) of a z-transform is the set of $z \in \mathbb{C}$ such that it converges
- The ROC is important
  - A z-transform converges for only certain values of z and does not exist for other values of z
  - z-transforms are non-unique without it
  - Always state the ROC when you state a z-transform
- The ROC is a connected annulus (doughnut) in the z-plane centered on the origin $z = 0$; that is, the ROC is of the form $r_1 < |z| < r_2$
  - If $x[n]$ has finite duration, then the ROC is the entire z-plane (except possibly $z = 0$ or $z = \infty$)
  - If $x[n]$ is causal, then the ROC is the outside of a disk
  - If $x[n]$ is anti-causal, then the ROC is the inside of a disk



z-Transform
Transfer Function
Poles and Zeros

Table of Contents

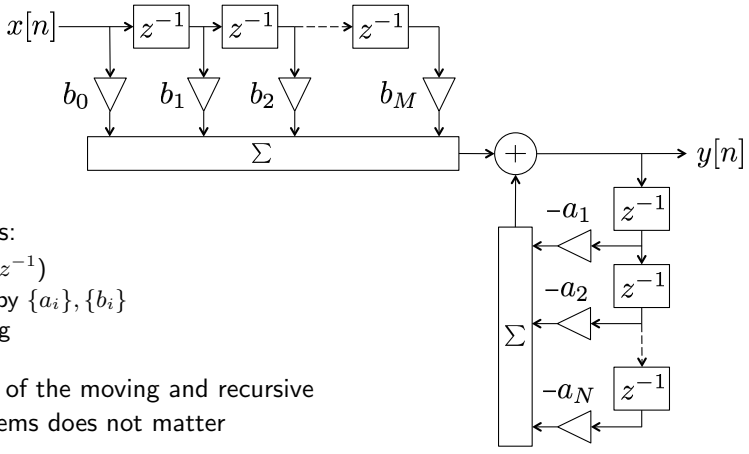
- Lecture in two parts:
  - Part 1: Transfer Function
  - Part 2: Poles and Zeros



z-Transform
Transfer Function

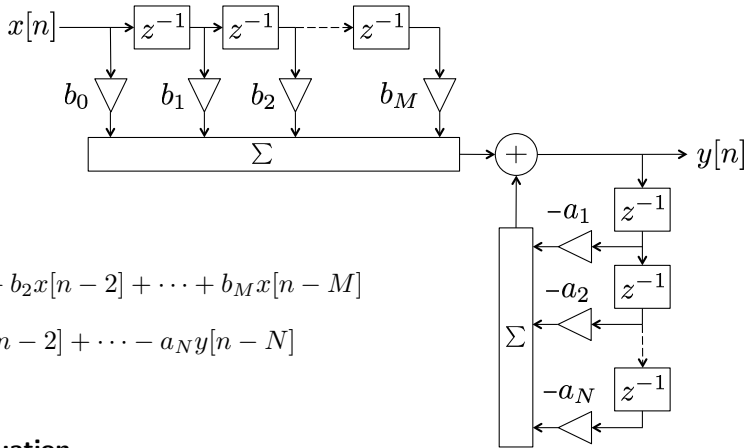
A General Class of LTI Systems

- The most general class of practical causal LTI systems (for infinite-length signals) consists of the cascade of a **moving average** system with a **recursive average** system



- Key elements:
  - Delays (z^{-1})
  - Scaling by $\{a_i\}, \{b_i\}$
  - Summing
- Note: Order of the moving and recursive average systems does not matter

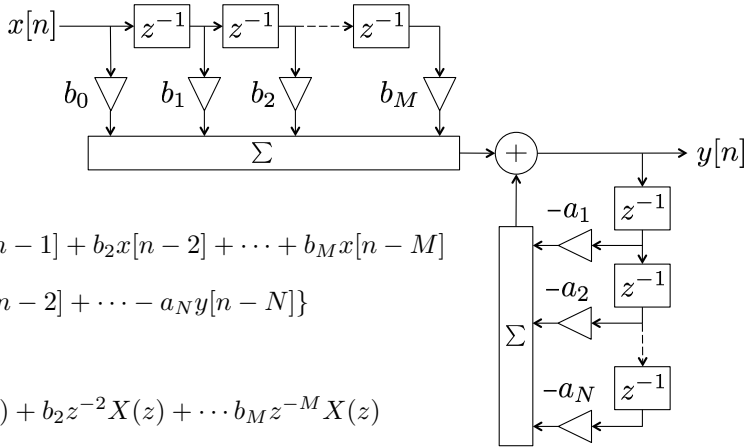
Input/Output Equation for General LTI System (Time Domain)



$$y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] + \cdots + b_Mx[n-M]$$
$$- a_1y[n-1] - a_2y[n-2] + \cdots - a_Ny[n-N]$$

This is called a **difference equation**
(Can process signals using the `filter` command in Matlab)

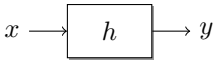
Input/Output Equation for General LTI System (z-Transform Domain)



$$\mathcal{Z}\{y[n]\} = \mathcal{Z}\{b_0x[n] + b_1x[n-1] + b_2x[n-2] + \cdots + b_Mx[n-M] - a_1y[n-1] - a_2y[n-2] + \cdots - a_Ny[n-N]\}$$

$$Y(z) = b_0X(z) + b_1z^{-1}X(z) + b_2z^{-2}X(z) + \cdots b_Mz^{-M}X(z) - a_1z^{-1}Y(z) - a_2z^{-2}Y(z) - \cdots - a_Nz^{-N}Y(z)$$

Transfer Function of an LTI System



$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- If

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad h[n] \overset{\mathcal{Z}}{\longleftrightarrow} H(z), \qquad y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$$

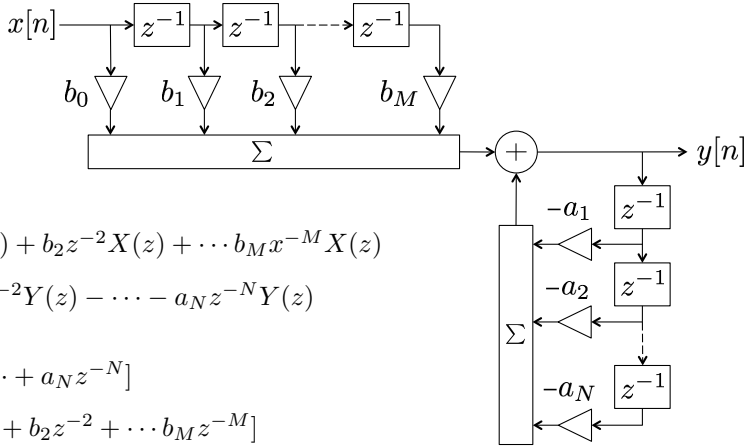
  then

$$Y(z) \;=\; H(z)\, X(z)$$

- **Transfer function**

$$H(z) \;=\; \frac{Y(z)}{X(z)} \;=\; \frac{\text{output}}{\text{input}}$$

Transfer Function of General LTI System



$$Y(z) = b_0X(z) + b_1z^{-1}X(z) + b_2z^{-2}X(z) + \cdots b_Mx^{-M}X(z)$$
$$- a_1z^{-1}Y(z) - a_2z^{-2}Y(z) - \cdots - a_Nz^{-N}Y(z)$$

$$Y(z)\left[1 + a_1z^{-1} + a_2z^{-2} + \cdots + a_Nz^{-N}\right]$$
$$= X(z)\left[b_0 + b_1z^{-1} + b_2z^{-2} + \cdots b_Mz^{-M}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1z^{-1} + b_2z^{-2} + \cdots b_Mz^{-M}}{1 + a_1z^{-1} + a_2z^{-2} + \cdots + a_Nz^{-N}}$$

Rational Transfer Functions

- The transfer function of the general LTI system is a **rational function** of polynomials in z^{-1}

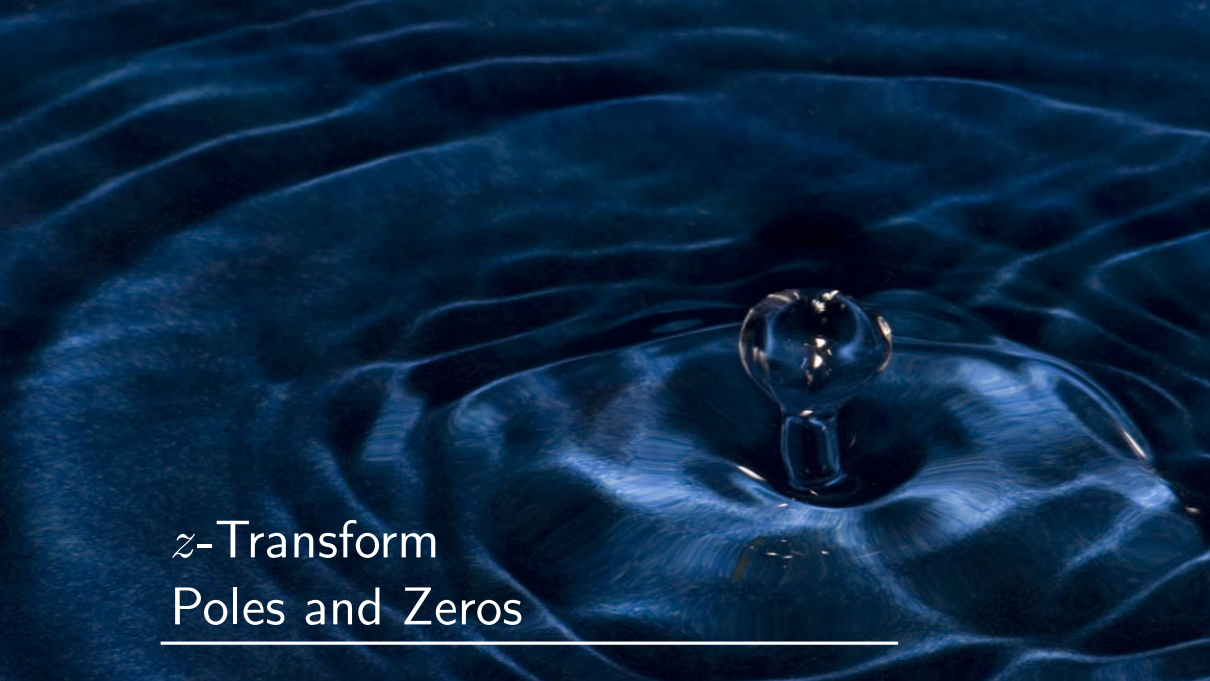
$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

  - **Numerator:** moving average part (non-recursive)
  - **Denominator:** recursive average part

- The **order** of the system is the larger of M and N

- Can easily convert $H(z)$ to positive powers of z by multiplying by $\frac{z^N}{z^M}\frac{z^M}{z^N} = 1$

$$H(z) = \frac{Y(z)}{X(z)} = \left(\frac{z^N}{z^M}\right) \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \cdots b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N}$$



z-Transform
Poles and Zeros

Poles and Zeros

- Factor the numerator and denominator polynomials (`roots` command in Matlab)

$$H(z) = \frac{Y(z)}{X(z)} = \left(\frac{z^N}{z^M}\right) \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \cdots b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N}$$

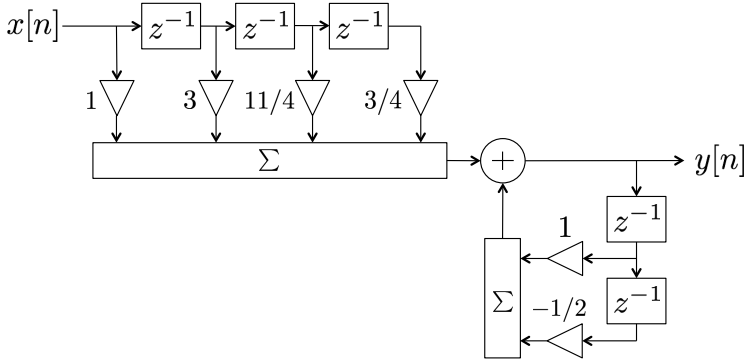
$$= \left(\frac{z^N}{z^M}\right) \frac{(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

- Roots of the numerator $\{\zeta_i\}$ are called **zeros**; at these values of z, $H(z) = 0$

- Roots of the denominator $\{p_i\}$ are called **poles**; at these values of z, $H(z) = \infty$

- Useful to plot the poles and zeros in the complex z-plane (`zplane` in Matlab)

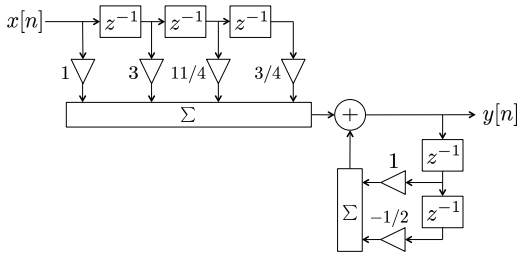
Example: Poles and Zeros (1)



- Given the block diagram, the transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \left(\frac{z^2}{z^3}\right) \frac{1\,z^3 + 3\,z^2 + 11/4\,z + 3/4}{z^2 - 1\,z + 1/2} = \frac{1 + 3\,z^{-1} + 11/4\,z^{-2} + 3/4\,z^{-3}}{1 - 1\,z^{-1} + 1/2\,z^{-2}}$$

Example: Poles and Zeros (2)



- Factor the transfer function into poles and zeros

$$H(z) = \frac{Y(z)}{X(z)} = \left(\frac{z^2}{z^3}\right) \frac{1\,z^3 + 3\,z^2 + 11/4\,z + 3/4}{z^2 - 1\,z + 1/2} = \frac{(z+1)(z+1/2)(z+3/2)}{z(z-1/2-j/2)(z-1/2+j/2)}$$

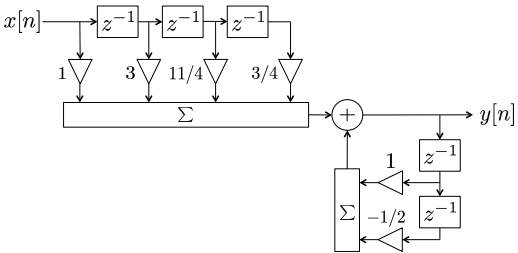
- Zeros at $z = -1/2,\ -1,\ -3/2$

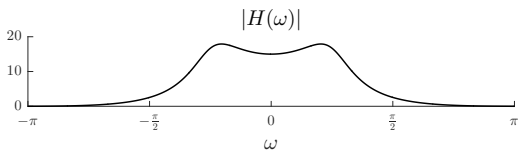
- Poles at $z = 0,\ 1/2 + j/2,\ 1/2 - j/2$
  Note: System is BIBO stable, since it is causal and all poles are inside the unit circle

Example: Poles and Zeros (3)

- We can obtain the **frequency response** by evaluating the transfer function $H(z)$ on the unit circle (`freqz` in Matlab)

$$H(e^{j\omega}) \;=\; \frac{1\,e^{j3\omega} + 3\,e^{j2\omega} + 11/4\,e^{j\omega} + 3/4}{e^{j\omega}(1\,e^{j2\omega} - 1\,e^{j\omega} + 1/2)}$$





Sketching $|H(z)|$ based on the Poles and Zeros

- Given the poles and zeros, there is an elegant geometrical interpretation of the value of $|H(z)|$

$$\begin{aligned} |H(z)| &= \left| \left(\frac{z^N}{z^M} \right) \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \cdots b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N} \right| \\ &= \left| \frac{z^N}{z^M} \right| \frac{|z - \zeta_1|\,|z - \zeta_2| \cdots |z - \zeta_M|}{|z - p_1|\,|z - p_2| \cdots |z - p_N|} \end{aligned}$$

- Note, for example, that

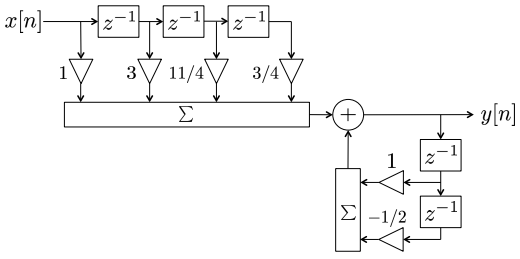
$$\begin{aligned} |z - p_1| &= \sqrt{[\mathrm{Re}(z) - \mathrm{Re}(p_1)]^2 + [\mathrm{Im}(z) - \mathrm{Im}(p_1)]^2} \\ &= \text{Euclidean distance between the points } z \text{ and } p_1 \text{ in the 2D complex } z \text{ plane} \end{aligned}$$

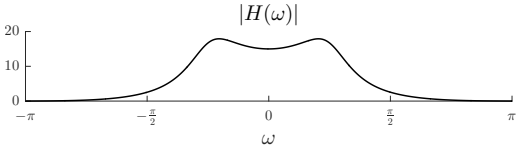
- Therefore

$$|H(z)| = \frac{\text{product of distances from } z \text{ to all zeros}}{\text{product of distances from } z \text{ to all poles}}$$

Example: Poles and Zeros (4)

$$H(z) = \frac{(z+1)(z+1/2)(z+3/2)}{z(z-1/2-j/2)(z-1/2+j/2)}$$

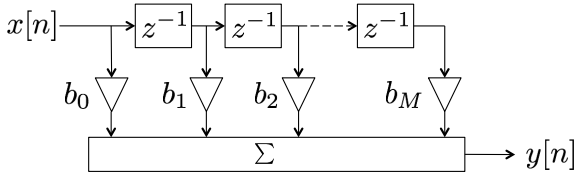




LTI System Design = Pole and Zero Placement

- Locations of the poles and zeros characterize an LTI system
- Hence, design of an LTI system is equivalent to designing where to place its poles and zeros

FIR Filters Have Only Zeros



- An FIR filter has **only zeros**

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_M z^{-M}$$
$$= (z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_M)$$

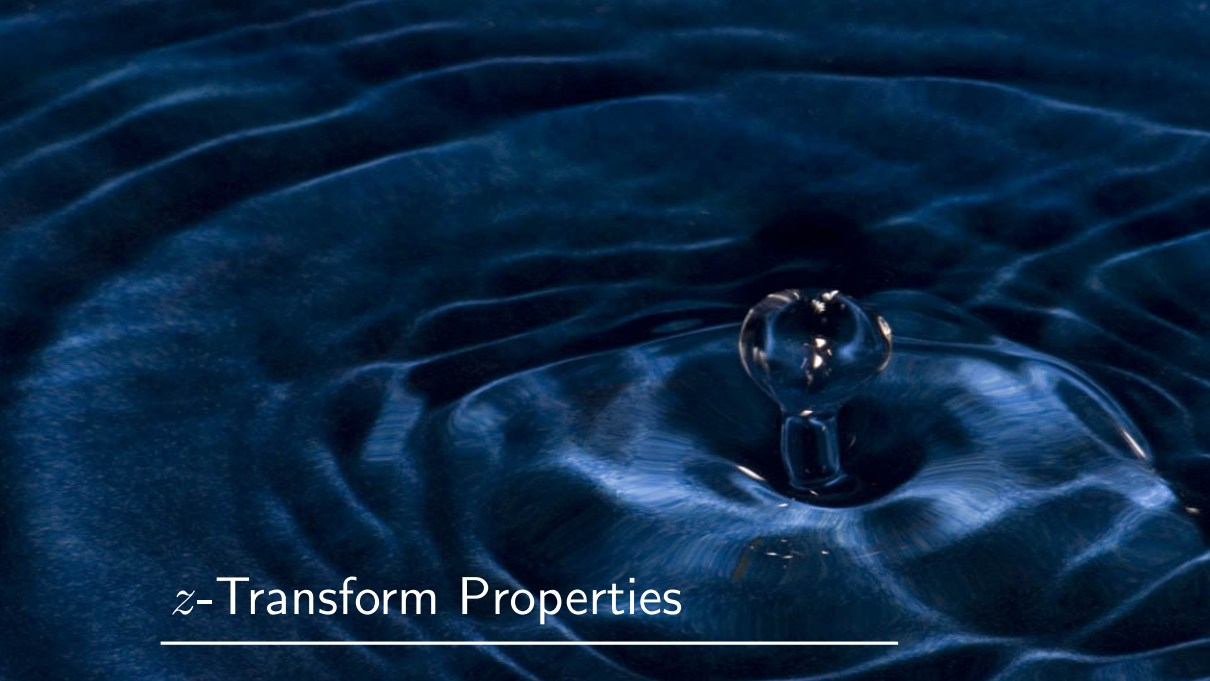
- Filter order = number of zeros = number of delays
- It is not uncommon to have hundreds or even thousands of zeros

Summary

- The most general class of practical causal LTI systems (for infinite-length signals) consists of the cascade of a **moving average** system with a **recursive average** system
- Transfer function $H(z) = \frac{Y(z)}{X(z)}$
- Zeros/poles $=$ roots of numerator/denominator of $H(z)$
- Elegant geometric formula

$$|H(z)| \;=\; \frac{\text{product of distances from } z \text{ to all zeros}}{\text{product of distances from } z \text{ to all poles}}$$

- Designing LTI systems is equivalent to placing their poles and zeros



z-Transform Properties

Recall: Forward z-Transform

- Forward z-transform of the signal $x[n]$ (analysis)

$$X(z) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}, \qquad z \in \mathbb{C},\; z \in \text{ROC}$$

- The z-transform is a complex function of a complex variable

- **z-transform pair**

$$x[n] \;\overset{\mathcal{Z}}{\longleftrightarrow}\; X(z)$$

z-Transform and DTFT

- Forward z-transform of the signal $x[n]$

$$X(z) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}, \qquad z \in \mathbb{C},\; z \in \text{ROC}$$

- DTFT of the signal $x[n]$

$$X(\omega) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad -\pi \leq \omega < \pi$$

- The DTFT is a special case of the z-transform (values on the unit circle in the complex z-plane)

$$X_{\text{DTFT}}(\omega) \;=\; X_z(z)|_{z=e^{j\omega}} \;=\; X_z(e^{j\omega})$$

z-Transform is Linear

- It is easy to show that if

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) \qquad x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z)$$

then

$$y[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z)$$

with

$$\mathsf{ROC}_Y = \mathsf{ROC}_{X_1} \cap \mathsf{ROC}_{X_2}$$

z-Transform and Time Shift

- If $x[n]$ and $X(z)$ are a z-transform pair then

$$x[n-m] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-m} X(z)$$

with no change to the ROC

- Proof: Use the change of variables $r = n - m$

$$\sum_{n=-\infty}^{\infty} x[n-m]\, z^{-n} = \sum_{r=-\infty}^{\infty} x[r]\, z^{-(r+m)} = \sum_{r=-\infty}^{\infty} x[r]\, z^{-r}\, z^{-m}$$

$$= z^{-m} \sum_{r=-\infty}^{\infty} x[r]\, z^{-r} = z^{-m}\, X(z) \quad \checkmark$$

z-Transform and Modulation

- If $x[n]$ and $X(z)$ are a z-transform pair then

$$z_0^n\, x[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad X(z/z_0)$$

- The ROC of $X(z/z_0)$ is $|z_0|\times$ROC of $X(z)$

- Proof:

$$\sum_{n=-\infty}^{\infty} z_0^n\, x[n]\, z^{-n} \;=\; \sum_{n=-\infty}^{\infty} x[n]\, (z/z_0)^{-n} \;=\; X(z/z_0) \;\checkmark$$

z-Transform and Conjugation

- If $x[n]$ and $X(z)$ are a DFT pair then

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$$

with no change to the ROC

- Proof:

$$\sum_{n=-\infty}^{\infty} x^*[n]\, z^{-n} = \left(\sum_{n=-\infty}^{\infty} x[n]\, (z^*)^{-n} \right)^* = X^*(z^*) \quad \checkmark$$

z-Transform and Time Reversal

- If $x[n]$ and $X(z)$ are a DFT pair then

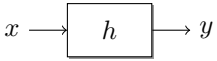
$$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z^{-1})$$

- The ROC is "inverted" in the sense that if the ROC of $X(z)$ is $r_1 < |z| < r_2$, then the ROC of $X(z^{-1})$ is $1/r_2 < |z| < 1/r_1$

- Proof: Use change of variables $m = -n$

$$\sum_{n=-\infty}^{\infty} x[-n]\, z^{-n} \;=\; \sum_{m=-\infty}^{\infty} x[m]\, z^{m} \;=\; \sum_{m=-\infty}^{\infty} x[m]\, (z^{-1})^{-m} \;=\; X(z^{-1}) \;\checkmark$$

z-Transform and Convolution



$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- If

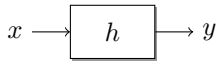
$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad h[n] \overset{\mathcal{Z}}{\longleftrightarrow} H(z), \qquad y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$$

  then

$$Y(z) \;=\; H(z)\, X(z), \qquad \mathsf{ROC}_Y = \mathsf{ROC}_X \cap \mathsf{ROC}_H$$

- Convolution in the time domain = multiplication in the z-transform domain

z-Transform, BIBO Stability, and Causality

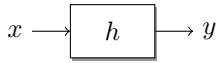


$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- Recall that an LTI system is BIBO stable iff $\|h\|_1 < \infty$

- **Fact:** An LTI system is BIBO stable iff the ROC of $H(z)$ includes the unit circle $|z| = 1$

Proof: z-Transform and BIBO Stability (1)

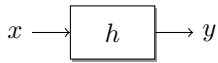


- Recall that the ROC of $H(z)$ is defined as the set of $z \in \mathbb{C}$ such that $h[n]\, z^{-n}$ is absolutely summable

$$S(z) \;=\; \sum_{n=-\infty}^{\infty} |h[n]|\, |z^{-n}| \;<\; \infty$$

- Suppose that the system is BIBO stable, which implies that $\|h\|_1 < \infty$. What can we say about the ROC?
  - When $|z| = 1$, we see that $S(z)|_{|z|=1} = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$ since $\|h\|_1 < \infty$
  - Thus, the unit circle $|z| = 1$ is in the ROC

Proof: z-Transform and BIBO Stability (1)

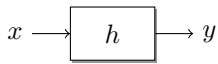


- Recall that the ROC of $H(z)$ is defined as the set of $z \in \mathbb{C}$ such that $h[n]\, z^{-n}$ is absolutely summable

$$S(z) \;=\; \sum_{n=-\infty}^{\infty} |h[n]|\, |z^{-n}| \;<\; \infty$$

- Suppose that the ROC of $H(z)$ includes the unit circle. What can we say about $h[n]$ and BIBO stability?
  - Since the ROC of $H(z)$ includes the unit circle, $S(1) < \infty$. But $S(1) = \sum_{n=-\infty}^{\infty} |h[n]| = \|h\|_1$
  - Thus, $\|h\|_1 < \infty$ and the system is BIBO stable

z-Transform, Poles, and BIBO Stability



- **Fact:** An LTI system is BIBO stable iff the ROC of $H(z)$ includes the unit circle $|z| = 1$

- **Corollary:** A **causal** LTI system is BIBO stable iff all of its poles are inside the unit circle

  - Since the systems is causal, the ROC extends outward from the pole that is furthest from the origin
  - Since the ROC contains the unit circle $|z| = 1$, the pole p furthest from the origin must have $|p| < 1$

Useful z-Transforms

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	$z \in \mathbb{C}$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$

For many more, see the supplementary materials, for example
`http://en.wikipedia.org/wiki/Z-transform`

Summary

- The z-transform has properties similar to the DTFT
- Convolution in time becomes multiplication in the z-transform domain
- An LTI system is BIBO stable iff the ROC of $H(z)$, the z-transform of the impulse response $h[n]$, includes the unit circle $|z| = 1$
- A **causal** LTI system is BIBO stable iff all of its poles are inside the unit circle

Inverse z-Transform

Recall: Forward z-Transform

- **Forward z-transform** of the signal $x[n]$ (analysis)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}, \qquad z \in \mathbb{C},\ z \in \text{ROC}$$

- Given $X(z)$, how to compute the **inverse z-transform** (synthesis)?

- **Complication:** $X(z)$ is a complex function of a complex variable z

Inverse z-Transform via Complex Integral

- Recall the **inverse DTFT**

$$x[n] \;=\; \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

- There exists a similar formula for the **inverse z-transform** via a contour integral in the complex z-plane

$$x[n] \;=\; \oint_{\mathcal{C}} X(z)\, z^{n}\, \frac{dz}{j2\pi z}$$

- Evaluation of such integrals is fun, but beyond the scope of this course

Inverse z-Transform via Factorization

- Any rational z-transform $X(z)$ can be factored into its zeros and poles

$$X(z) = \frac{(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)} = z^{N-M}\frac{(1-\zeta_1 z^{-1})(1-\zeta_2 z^{-1})\cdots(1-\zeta_M z^{-1})}{(1-p_1 z^{-1})(1-p_2 z^{-1})\cdots(1-p_N z^{-1})}$$

- Inverting the z-transform for one pole $\frac{1}{1-p_1 z^{-1}}$ is easy (assume causal inverse)

$$p_1^n\, u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-p_1 z^{-1}}$$

- Fortunately, there is a method to decompose $X(z)$ into a sum, rather than a product, of poles

$$X(z) = \frac{C_1}{1-p_1 z^{-1}} + \frac{C_2}{1-p_2 z^{-1}} + \cdots + \frac{C_N}{1-p_N z^{-1}}$$

- Then we can just invert the z-transform of each term separately and sum the results

Partial Fraction Expansion

- The **partial fraction expansion** decomposes a rational function of z^{-1}

$$X(z) \;=\; \frac{\text{polynomial in } z^{-1}}{(1-p_1 z^{-1})(1-p_2 z^{-1})\cdots(1-p_N z^{-1})}$$

  into a sum of terms, one for each pole

$$X(z) \;=\; \frac{C_1}{1-p_1 z^{-1}} \;+\; \frac{C_2}{1-p_2 z^{-1}} \;+\; \cdots \;+\; \frac{C_N}{1-p_N z^{-1}}$$

- There are many algorithms to compute a partial fraction expansion; here we introduce one of the simplest using a running example

- Note: We will explain partial fractions only for non-repeated poles; see the supplementary material for the case when there are repeated poles (not difficult!)

Partial Fraction Expansion Algorithm – Step 1

- Goal: Invert the z-transform of the rational function

$$X(z) = \frac{5 + 2z^{-1} - z^{-2}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$$

using the partial fraction expansion

- **Step 1:** Check if (order of denominator polynomial) $>$ (order of numerator polynomial); otherwise perform polynomial division to make it so
  - Not true in our case, so use polynomial long division to divide the denominator into the numerator

$$X(z) = 6 + \frac{-1 + 3z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = 6 + X'(z)$$

We will work on $X'(z)$ and then recombine it with the 6 at the end

Partial Fraction Expansion Algorithm – Step 2

- **Step 2:** Factor the denominator polynomial into poles (no need to factor the numerator)

$$X'(z) = \frac{-1+3z^{-1}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}} = \frac{-1+3z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}$$

Partial Fraction Expansion Algorithm – Step 3

- **Step 3:** Assuming that no poles are repeated (same location in the z-plane), break $X'(z)$ into a sum of terms, one for each pole

$$X'(z) = \frac{-1+3z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} = \frac{C_1}{1-\frac{1}{2}z^{-1}} + \frac{C_2}{1+\frac{1}{3}z^{-1}}$$

- Note: Repeated poles just require that we add extra terms to the sum. The details are not difficult – see the Supplementary Resources

Partial Fraction Expansion Algorithm – Step 4

- **Step 4:** To determine C_1 and C_2, bring the sum of terms back to a common denominator and compare its terms to terms of the same power in $X'(z)$

$$X'(z) = \frac{-1+3z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} = \frac{C_1}{1-\frac{1}{2}z^{-1}} + \frac{C_2}{1+\frac{1}{3}z^{-1}}$$

$$= \frac{C_1(1+\frac{1}{3}z^{-1}) + C_2(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} = \frac{(C_1+C_2) + (\frac{C_1}{3} - \frac{C_2}{2})z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}$$

- This yields the set of 2 simultaneous linear equations

$$\text{powers of } 1: \quad -1 = C_1 + C_2$$

$$\text{powers of } z^{-1}: \quad 3 = \frac{C_1}{3} - \frac{C_2}{2}$$

- Solve by your favorite method to obtain: $C_1 = 3, \ C_2 = -4$

Partial Fraction Expansion Algorithm – Step 5

- **Final partial fractions result**

$$X(z) \;=\; 6 \;+\; \frac{3}{1-\frac{1}{2}z^{-1}} \;+\; \frac{-4}{1+\frac{1}{3}z^{-1}}$$

- **Step 5:** Check your work by bringing the partial fractions result to a common demoninator and making sure it agrees with what you started with

$$\begin{aligned} X(z) &= 6 + \frac{3}{1-\frac{1}{2}z^{-1}} + \frac{-4}{1+\frac{1}{3}z^{-1}} \\ &= \frac{6(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1}) + 3(1+\frac{1}{3}z^{-1}) - 4(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} \\ &= \frac{5+2z^{-1}-z^{-2}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}} \quad \checkmark \end{aligned}$$

Back To Our Regularly Scheduled Program . . .

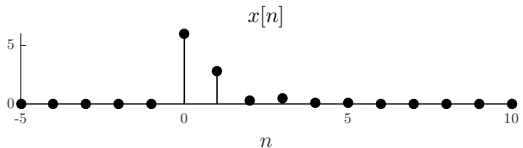
- Thanks to the partial fractions expansion, we can write

$$X(z) \;=\; \frac{5+2z^{-1}-z^{-2}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}} \;=\; 6 \;+\; \frac{3}{(1-\frac{1}{2}z^{-1})} \;+\; \frac{-4}{(1+\frac{1}{3}z^{-1})}$$

  and so it is easy to invert the z-transform $X(z)$ term-by-term (thanks to linearity)

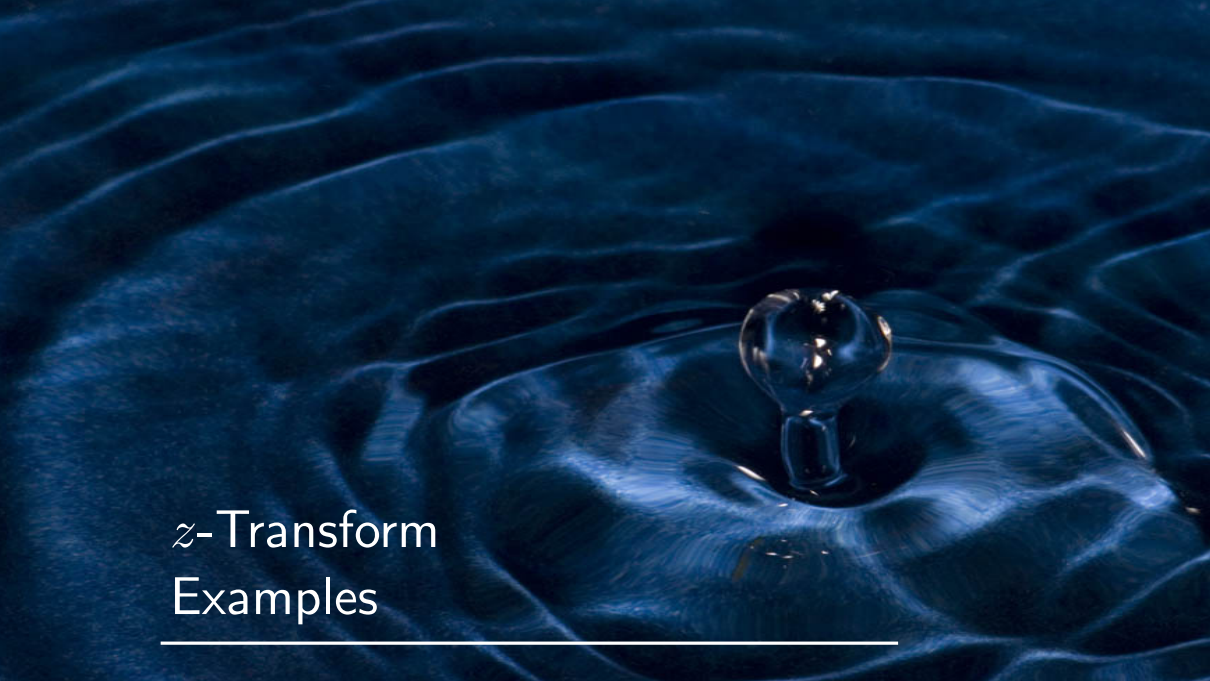
- Assuming that $x[n]$ is causal (ROC: $|z| > \frac{1}{2}$) yields

$$x[n] \;=\; 6\,\delta[n] \;+\; 3\left(\frac{1}{2}\right)^n u[n] \;-\; 4\left(\frac{-1}{3}\right)^n u[n]$$



Summary

- Inverse z-transform is more complicated to compute than inverse Fourier transforms
- In practice, we invert the z-transform using the partial fraction expansion and the inverting term-by-term
- Partial fraction expansion is a tedious process, but straightforward (and useful!)

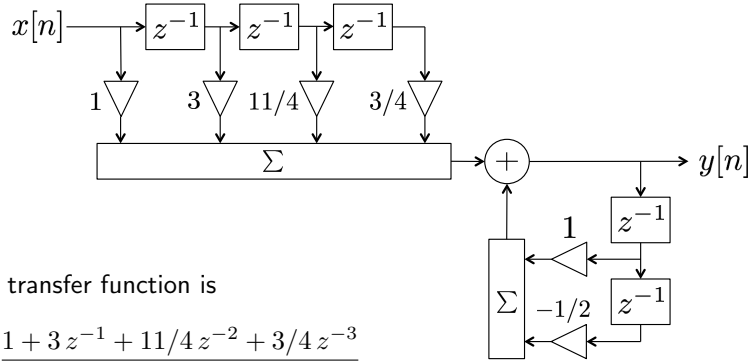


z-Transform Examples

z-Transform Tools in Matlab

- There are many useful commands in Matlab related to the z-transform. **Click here** to view a video demonstration.
  - `roots` – to factor polynomials and find zeros and poles
  - `zplane` – to plot poles and zeros in the complex z-plane
  - `freqz` – to plot the DTFT frequency response corresponding to a z transform transfer function
  - `filter` – to process an input signal and produce an output signal
  - ...

Example LTI System



- Given the block diagram, the transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 3\,z^{-1} + 11/4\,z^{-2} + 3/4\,z^{-3}}{1 - 1\,z^{-1} + 1/2\,z^{-2}}$$

- Matlab encodes the coefficients according to two vectors

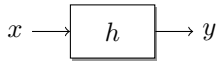
$$a = \begin{bmatrix} 1 & -1 & 1/2 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & 3 & 11/4 & 3/4 \end{bmatrix}$$

Discrete-Time Filters

Set H

Discrete-Time Filters

Putting LTI Systems to Work



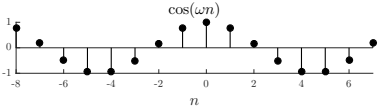
$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

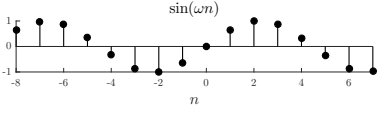
$$Y(z) \;=\; H(z)\, X(z), \qquad Y(\omega) \;=\; H(\omega)\, X(\omega)$$

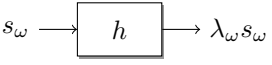
- **Goal:** Design a LTI system to perform a certain **task** in some application
- Key questions:
  - What is the **range of tasks** that an LTI system can perform?
  - What are the **parameters** under our control for design purposes?

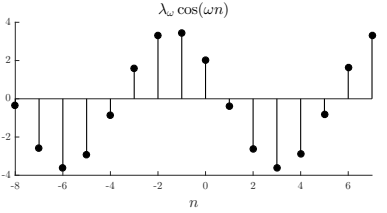
What Do LTI Systems Do? Recall Eigenanalysis

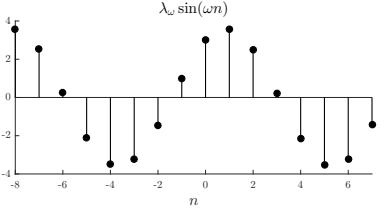
- LTI system **eigenvectors**: $s_\omega[n] = e^{j\omega n}$
- LTI system **eigenvalues**: $\lambda_\omega = H(\omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$ (frequency response)



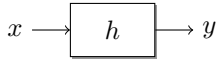








LTI Systems Filter Signals



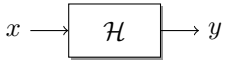
- Important interpretation of $Y(\omega) = H(\omega)\, X(\omega)$

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi} \longrightarrow \boxed{h} \longrightarrow y[n] = \int_{-\pi}^{\pi} H(\omega)\, X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

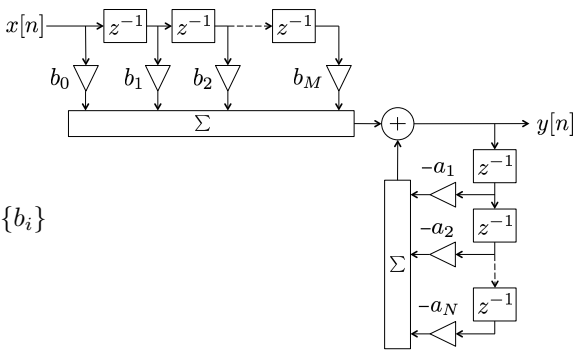
- An LTI system processes a signal $x[n]$ by amplifying or attenuating the sinusoids in its Fourier representation (DTFT) $X(\omega)$ by the complex factor $H(\omega)$

- Inspires the terminology that $X(\omega)$ is **filtered** by $H(\omega)$ to produce $Y(\omega)$

Design Parameters of Discrete-Time Filters (LTI Systems)

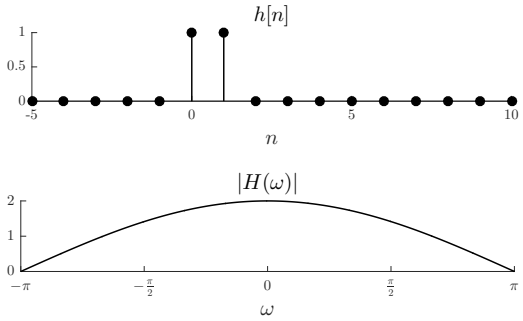


- Impulse response: $h[n]$
- Transfer function: $H(z)$
  - poles and zeros
- Frequency response: $H(\omega)$
- Moving/recursive average parameters: $\{a_i\}$, $\{b_i\}$



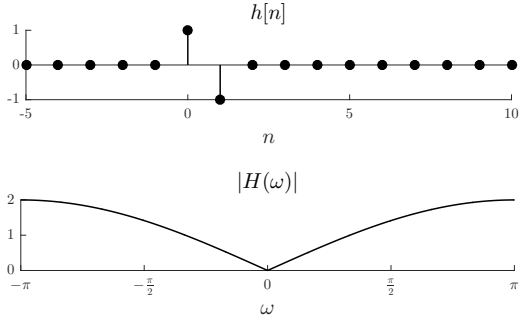
Filters Archetypes: Low-Pass

- Ideal low-pass filter
- Example low-pass impulse response $h[n]$
- Example frequency response $|H(\omega)|$



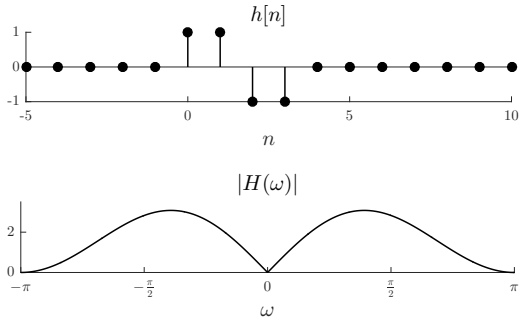
Filters Archetypes: High-Pass

- Ideal high-pass filter
- Example high-pass impulse response $h[n]$
- Example frequency response $|H(\omega)|$



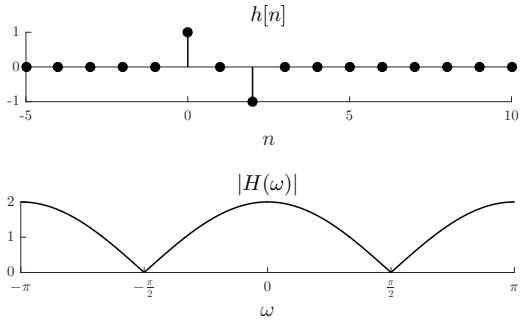
Filters Archetypes: Band-Pass

- Ideal band-pass filter
- Example band-pass impulse response $h[n]$
- Example frequency response $|H(\omega)|$



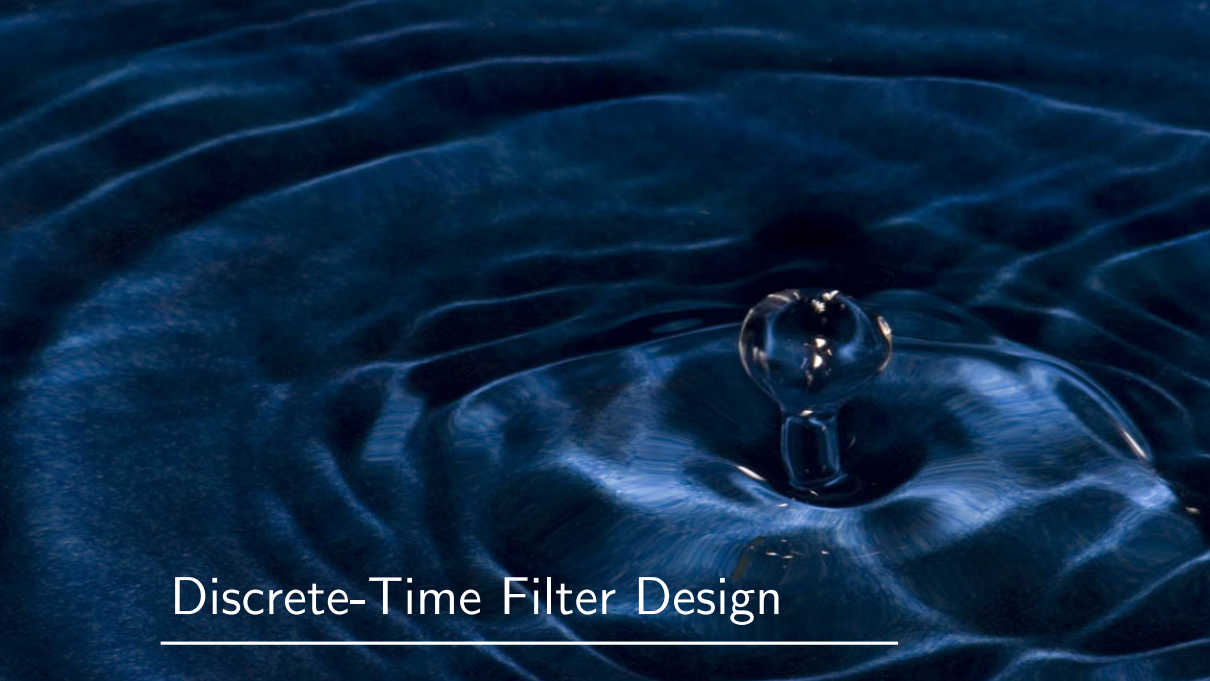
Filters Archetypes: Band-Stop

- Ideal band-stop filter
- Example band-stop impulse response $h[n]$
- Example frequency response $|H(\omega)|$



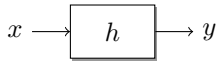
Summary

- Now that we understand what LTI systems do, we can **design** them to accomplish certain tasks
- An LTI system processes a signal $x[n]$ by amplifying or attenuating the sinusoids in its Fourier representation (DTFT)
- Equivalent design parameters of a discrete-time filter
  - Impulse response: $h[n]$
  - Transfer function: $H(z)$ (poles and zeros)
  - Frequency response: $H(\omega)$
  - Moving/recursive average parameters: $\{a_i\}$, $\{b_i\}$
- Archetype filters: Low-pass, high-pass, band-pass, band-stop
- We will emphasize infinite-length signals, but the situation is similar for finite-length signals



Discrete-Time Filter Design

Recall Discrete-Time Filter



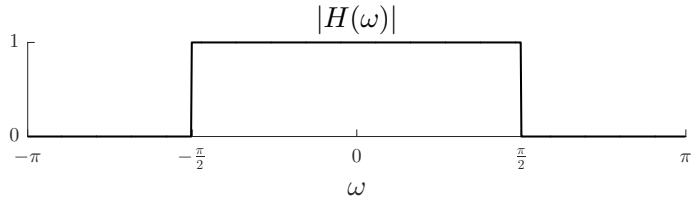
$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

$$Y(z) \;=\; H(z)\, X(z), \qquad Y(\omega) \;=\; H(\omega)\, X(\omega)$$

- A discrete-time filter fiddles with a signal's Fourier representation
- Recall the filter archetypes: ideal low-pass, high-pass, band-pass, band-stop filters

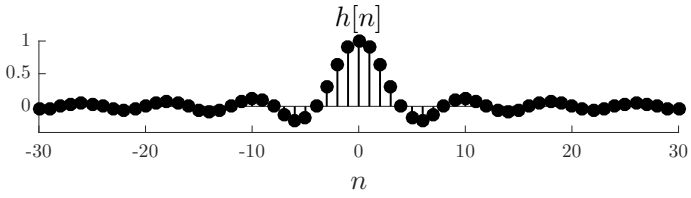
Ideal Lowpass Filter

- Ideal low-pass filter frequency response $H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$



Ideal Lowpass Filter

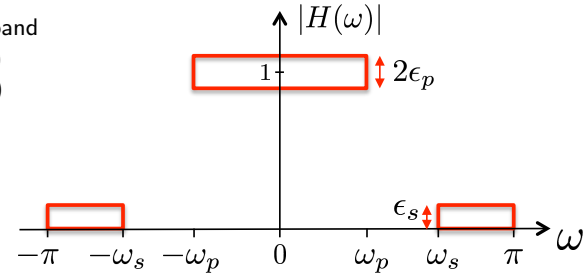
- Ideal low-pass filter frequency response $H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$
- Impulse response is the infamous "sinc" function $h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$



- Problems:
  - System is not BIBO stable! $(\sum_n |h[n]| = \infty)$
  - Infinite computational complexity ($H(z)$ is not a rational function)

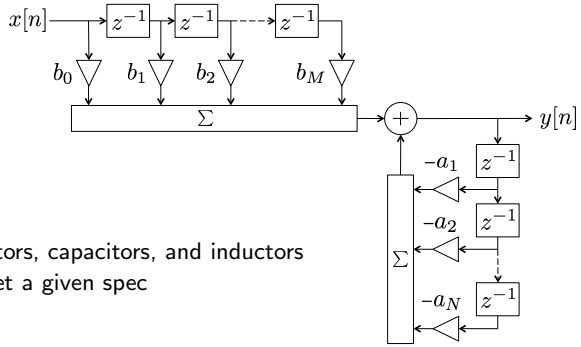
Filter Specification

- Find a filter of minimum complexity that meets a given **specification**
- Example: Low-pass filter
  - Pass-band edge frequency: ω_p
  - Stop-band edge frequency: ω_s
  - Between pass- and stop-bands: transition band
  - Pass-band ripple ϵ_p (often expressed in dB)
  - Stop-band ripple ϵ_s (often expressed in dB)



- Clearly, the tighter the specs, the more complex the filter

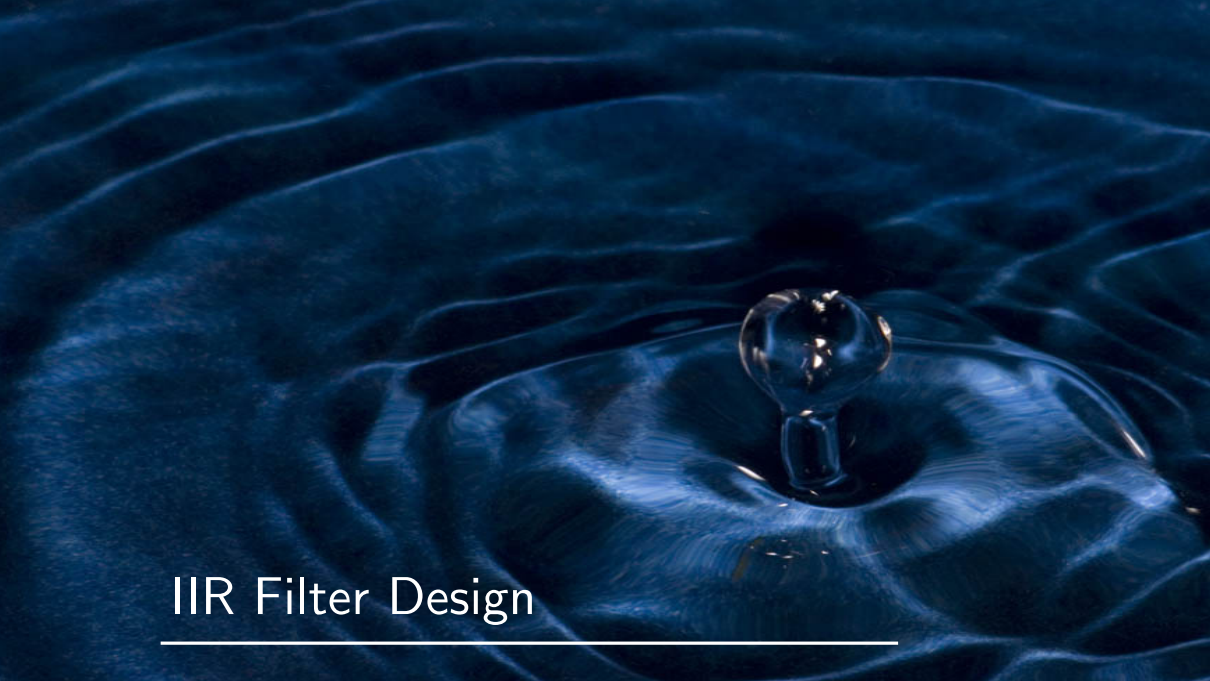
Two Classes of Discrete-Time Filters



- Infinite impulse response (IIR) filters
  - Uses both moving and recursive averages
  - $H(z)$ has both poles and zeros
  - Related to "analog" filter design using resistors, capacitors, and inductors
  - Generally have the lowest complexity to meet a given spec
- Finite impulse response (FIR) filters
  - Uses only moving average
  - $H(z)$ has only zeros
  - Unachievable in analog using resistors, capacitors, and inductors
  - Generally higher complexity (than IIR) to meet a given spec
  - But can have linear phase (big plus)

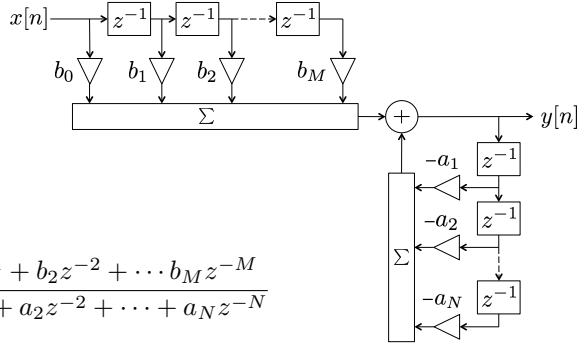
Summary

- A discrete-time filter fiddles with a signal's Fourier representation
- "Ideal" filters are not practical
  - System is not BIBO stable!
  - Infinite computational complexity ($H(z)$ is not a rational function)
- Filter design: Find a filter of minimum complexity that meets a given spec
- Two different types of filters (IIR, FIR) mean two different types of filter design



IIR Filter Design

IIR Filters



- Use both moving and recursive averages
- Transfer function has both **poles** and **zeros**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$$= z^{N-M} \frac{(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

- We **design** an IIR filter by specifying the locations of its poles and zeros in the z-plane
- Generally can satisfy a spec with lower complexity than FIR filters

IIR Filters from Analog Filters

- In contrast to FIR filter design, IIR filters are typically designed by a two-step procedure that is slightly ad hoc

- **Step 1:** Design an **analog** filter (for resistors, capacitors, and inductors) using the Laplace transform $H_L(s)$ (this theory is well established but well beyond the scope of this course)

- **Step 2:** Transform the analog filter into a discrete-time filter using the **bilinear transform** (a conformal map from complex analysis)

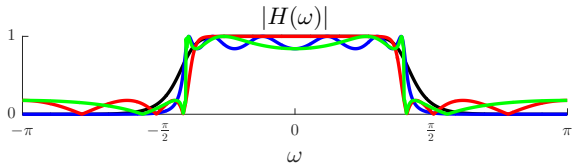
$$s \;=\; c\,\frac{z-1}{z+1}$$

- The discrete-time filter's transfer function is given by

$$H(z) \;=\; H_L(s)\big|_{s=c\frac{z-1}{z+1}}$$

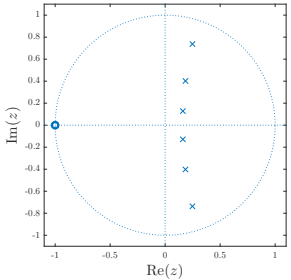
Three Important Classes of IIR Filters

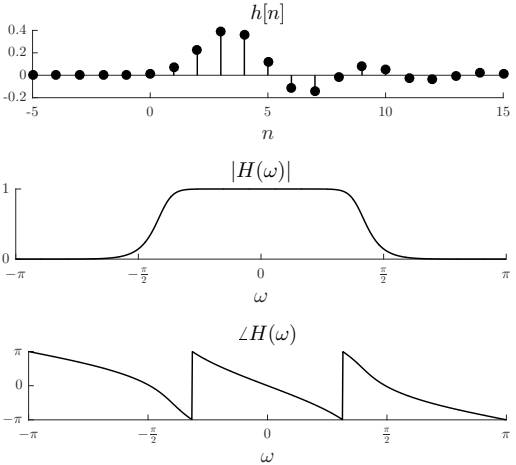
- Butterworth filters
  - `butter` command in Matlab
  - No ripples (oscillations) in $|H(\omega)|$
  - Gentlest transition from pass-band to stop-band for a given order
- Chebyshev filters
  - `cheby1` and `cheby2` commands in Matlab
  - Ripples in either pass-band or stop-band
- Elliptic filters
  - `ellip` command in Matlab
  - Ripples in both pass-band and stop-band
  - Sharpest transition from pass-band to stop-band for a given order
    (use with caution!)



Butterworth IIR Filter

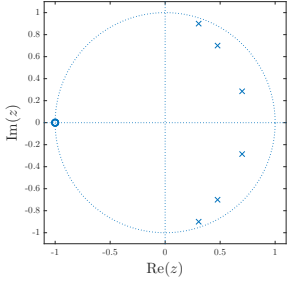
- "Maximally flat" frequency response
  - Largest number of derivatives of $|H(\omega)|$ equal to 0 at $\omega = 0$ and π
- N zeros and N poles
  - Zeros are all at $z = -1$
  - Poles are located on a circle inside the unit circle
- Example: $N = 6$ using `butter` command in Matlab

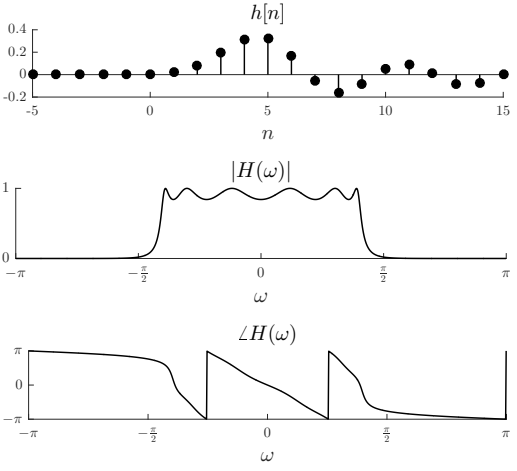




Chebyshev Type 1 IIR Filter

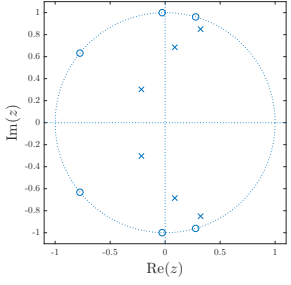
- Ripples/oscillations (of equal amplitude) in the pass-band and not in the stop-band
- N zeros and N poles
  - Zeros are all at $z = -1$
  - Poles are located on an ellipse inside the unit circle
- Example: $N = 6$ using `cheby1` command in Matlab

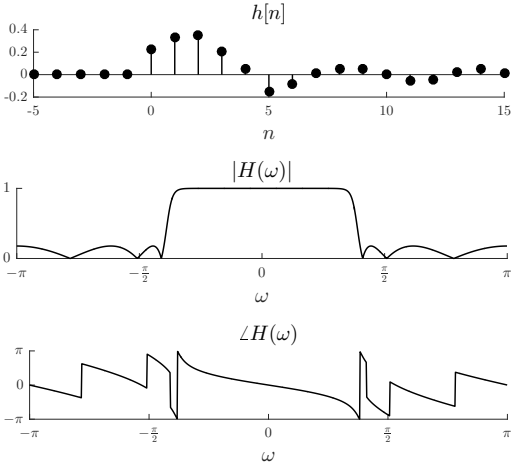




Chebyshev Type 2 IIR Filter

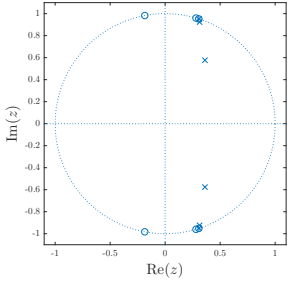
- Ripples/oscillations (of equal amplitude) in the stop-band and not in the pass-band
- N zeros and N poles
  - Zeros are distributed on unit circle
  - Poles are located on an ellipse inside the unit circle
- Example: $N = 6$ using `cheby2` command in Matlab

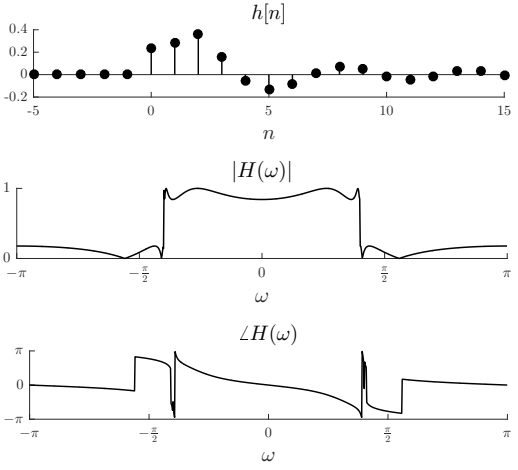




Elliptic IIR Filter

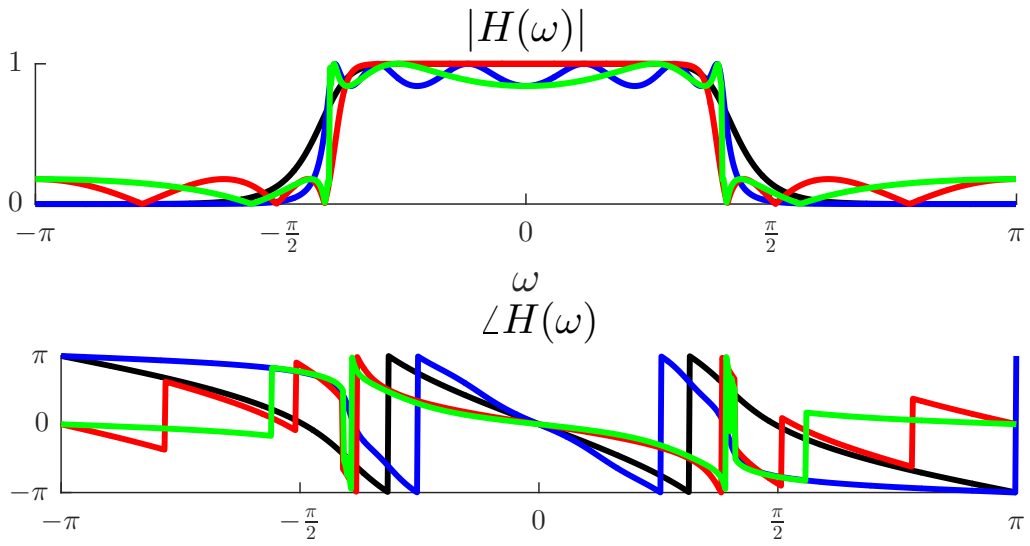
- Ripples/oscillations in both the stop-band and pass-band
- N zeros and N poles
  - Zeros are clustered on unit circle near ω_p
  - Poles are clustered close to unit circle near ω_p
- Example: $N = 6$ using `ellip` command in Matlab





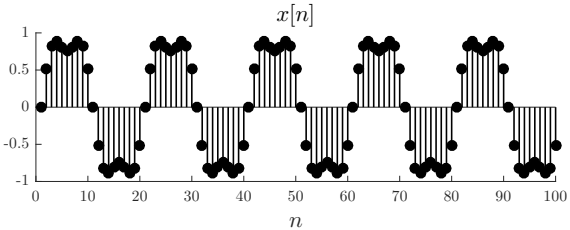
IIR Filter Comparison

- Butterworth (black), Chebyshev 1 (blue), Chebyshev 2 (red), Elliptic (green)

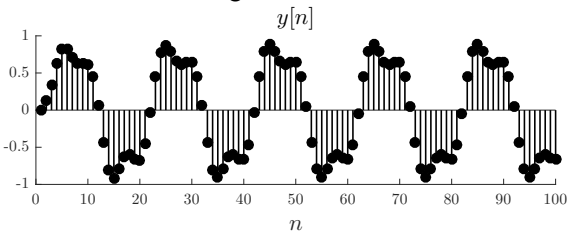


IIR Filter Phase Shift

- Note how the elliptic filter's phase response is non-linear across the passband.
- Non-linear phase shifts introduces distortion. Consider signal $x[n]$, a sum of two sinusoids:

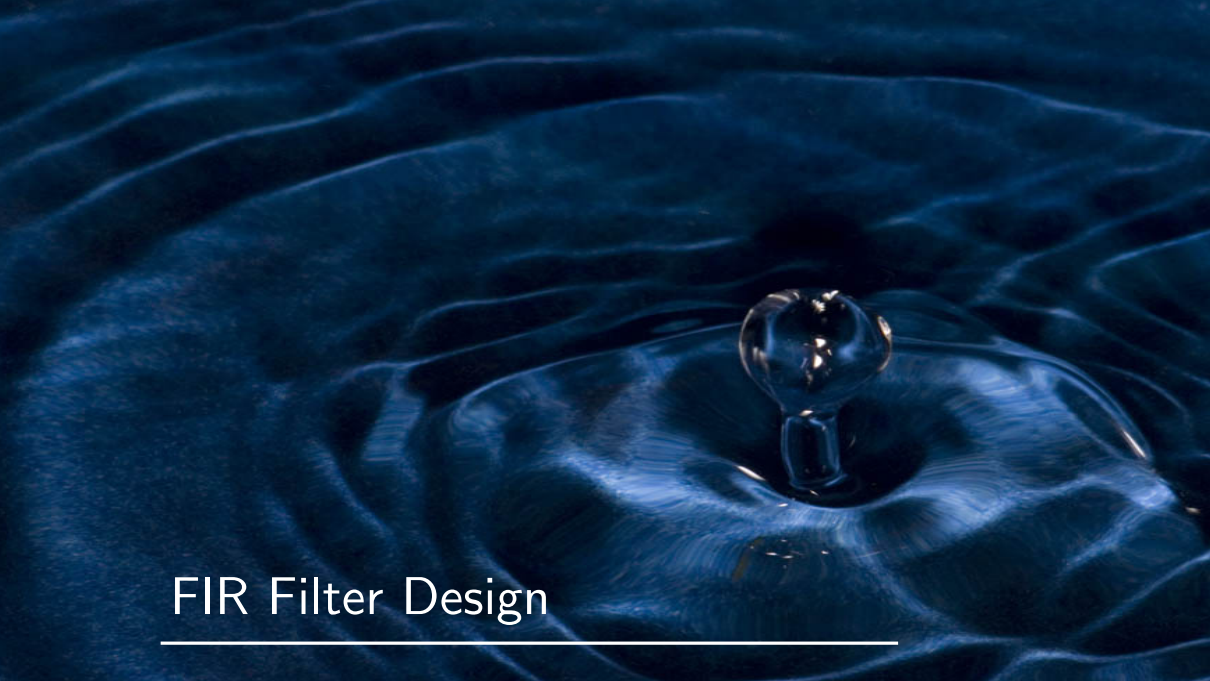


- Both are in the pass-band, so the output $y[n]$ should be the same as the input. But Non-linear phase shift puts the two sinusoids out of alignment:



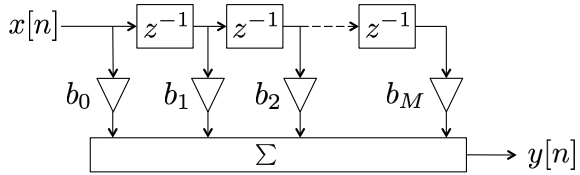
Summary

- IIR filters use use both moving and recursive averages and have both poles and zeros
- Typically designed by transforming an analog filter design (for use with resistors, capacitors, and inductors) into discrete-time via the bilinear transform
- Four families of IIR filters: Butterworth, Chebyshev (1,2), Elliptic
- Useful Matlab commands for choosing the filter order N that meets a given spec: `butterord`, `cheby1ord`, `cheby2ord`, `ellipord`



FIR Filter Design

FIR Filters



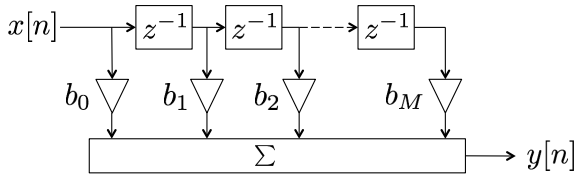
- Use only a moving average
- Transfer function has only **zeros** (and trivial poles at $z = 0$)

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_M z^{-M}$$

$$= z^{-M}(z - \zeta_1)(z - \zeta_2)\cdots(z - \zeta_M)$$

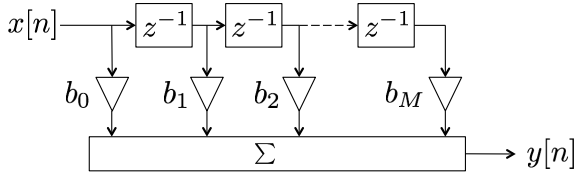
- We **design** an FIR filter by specifying the values of the **taps** $b_0, b_1, \ldots, b_M$ (this is equivalent to specifying the locations of the zeros in the z-plane)
- Generally require a higher complexity to meet a given spec than an IIR filter

FIR Filters Are Interesting



- FIR filters are **specific to discrete-time**; they cannot be built in analog using R, L, C
- FIR filters are always BIBO stable
- FIR filters can be designed to **optimally** meet a given spec
- Unlike IIR filters and all analog filters, FIR filters can have (generalized) **linear phase**
  - A nonlinear phase response $\angle H(\omega)$ distorts signals as they pass through the filter
  - Recall that a linear phase shift in the DTFT is equivalent to a simple time shift in the time domain

Impulse Response of an FIR Filter



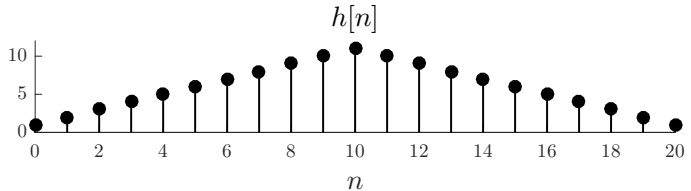
- Easy to see by inputting $x[n] = \delta[n]$ that
  the **impulse response** of an FIR filter consists of the taps weights

n	$\cdots$	-2	-1	0	1	2	$\cdots$	M	$M+1$	$M+2$	$\cdots$
$h[n]$	$\cdots$	0	0	b_0	b_1	b_2	$\cdots$	b_M	0	0	$\cdots$

- Note: Filter **order** $= M$; filter **length** $= M + 1$

Symmetric FIR Filters

- Unlike IIR filters, FIR filters can be causal and have (generalized) linear phase
- Linear phase filters must have a symmetric impulse response
  - Four cases: even/odd length, even/odd symmetry
  - Different symmetries can be useful for different filter types (low-pass, high-pass, etc.)
- We will focus here on low-pass filters with **odd-length**, **even-symmetric** impulse response
  - Odd length: $M+1$ is odd (M is even)
  - Even symmetric (around the center of the filter): $h[n] = h[M-n], \;\; n = 0, 1, \ldots, M$
- Example: Length $M+1 = 21$



Frequency Response of a Symmetric FIR Filter (1)

- Compute frequency response when $h[n]$ is **odd-length** and **even-symmetric** ($h[n] = h[M-n]$)

$$
\begin{aligned}
H(\omega) &= \sum_{n=0}^{M} h[n]\, e^{-j\omega n} = \sum_{n=0}^{M/2-1} h[n]\, e^{-j\omega n} + h[M/2]\, e^{-j\omega M/2} + \sum_{n=M/2+1}^{M} h[n]\, e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]\, e^{-j\omega n} + h[M/2]\, e^{-j\omega M/2} + \sum_{n=M/2+1}^{M} h[M-n]\, e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]\, e^{-j\omega n} + h[M/2]\, e^{-j\omega M/2} + \sum_{r=0}^{M/2-1} h[r]\, e^{-j\omega(M-r)} \\
&= h[M/2]\, e^{-j\omega M/2} + \sum_{n=0}^{M/2-1} h[n] \left(e^{-j\omega n} + e^{j\omega(n-M)} \right)
\end{aligned}
$$

Frequency Response of a Symmetric FIR Filter (2)

- Compute frequency response when $h[n]$ is **odd-length** and **even-symmetric** ($h[n] = h[M-n]$)

$$
\begin{aligned}
H(\omega) &= h[M/2]\, e^{-j\omega M/2} + \sum_{n=0}^{M/2-1} h[n] \left(e^{-j\omega n} + e^{j\omega(n-M)} \right) \\
&= h[M/2]\, e^{-j\omega M/2} + \sum_{n=0}^{M/2-1} h[n]\, e^{-j\omega M/2} \left(e^{-j\omega(n-M/2)} + e^{j\omega(n-M/2)} \right) \\
&= \left(h[M/2] + \sum_{n=0}^{M/2-1} 2\, h[n] \cos(\omega(n - M/2)) \right) e^{-j\omega M/2} \\
&= A(\omega)\, e^{-j\omega M/2}
\end{aligned}
$$

Generalized Linear Phase FIR Filters

- Frequency response when $h[n]$ is **odd-length** and **even-symmetric** ($h[n] = h[M-n]$)

$$H(\omega) \;=\; A(\omega)\, e^{-j\omega M/2}$$

with

$$A(\omega) \;=\; h[M/2] \;+\; \sum_{n=0}^{M/2-1} 2\, h[n] \cos(\omega(n - M/2))$$

- $A(\omega)$ is called the **amplitude** of the filter; it plays a role like $|H(\omega)|$ since

$$|H(\omega)| \;=\; |A(\omega)|$$

However, $A(\omega)$ is not necessarily ≥ 0

- $e^{-j\omega M/2}$ is a **linear phase shift**
$H(\omega)$ has linear phase except when $A(\omega)$ changes sign, in which case its phase jumps by π rad

FIR Filter Design

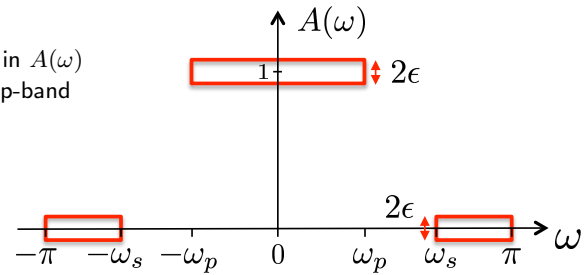
- Frequency response when $h[n]$ is **odd-length** and **even-symmetric** ($h[n] = h[M-n]$)

$$H(\omega) \;=\; A(\omega)\, e^{-j\omega M/2}$$

with

$$A(\omega) \;=\; h[M/2] \;+\; \sum_{n=0}^{M/2-1} 2\, h[n] \cos(\omega(n - M/2))$$

- Design of $H(\omega)$ is equivalent to the design of $A(\omega)$; spec changes slightly
  - Stop-band spec now allows negative values in $A(\omega)$
  - For simplicity, same ϵ in both pass- and stop-band (this is easy to generalize)



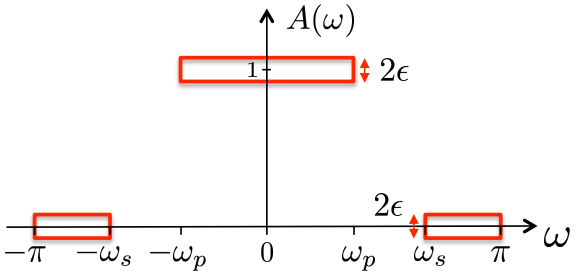
Optimal FIR Filter Design

- **Goal:** Find the **optimal** $A(\omega)$ (in terms of shortest length $M+1$) that meets the specs

$$A(\omega) \;=\; h[M/2] \;+\; \sum_{n=0}^{M/2-1} 2\,h[n]\cos(\omega(n-M/2))$$

- Parameters under our control: The $M/2+1$ filter taps $h[n]$, $n = 0, 1, \ldots, M/2$

- Problem solved by James McClellan and Thomas Parks at Rice University (1971) "Parks-McClellan Filter Design"

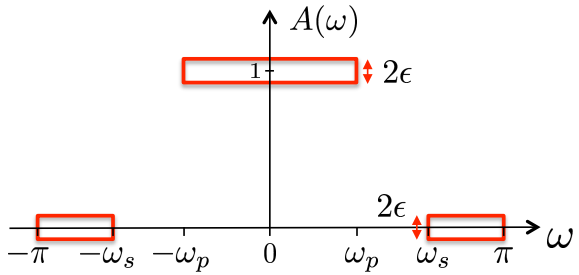


Key Ingredients of Optimal FIR Filter Design

- **Goal:** Find the **optimal** $A(\omega)$ (in terms of shortest length $M+1$) that meets the specs

$$A(\omega) \;=\; h[M/2] \;+\; \sum_{n=0}^{M/2-1} 2\,h[n]\cos(\omega(n-M/2))$$

- **Ripples:** $A(\omega)$ oscillates $M/2$ times in the interval $0 \le \omega \le \pi$
- **Equiripple property:** The oscillations of the optimal $A(\omega)$ are all the same size
- **Alternation Theorem:** The optimal $A(\omega)$ will touch the error bounds $M/2+2$ times in the interval $0 \le \omega \le \pi$



Remez Exchange Algorithm for Optimal FIR Filter Design

- **Goal:** Find the **optimal** $A(\omega)$ (in terms of shortest length $M+1$) that meets the specs

$$A(\omega) \;=\; h[M/2] \;+\; \sum_{n=0}^{M/2-1} 2\,h[n]\cos(\omega(n-M/2))$$

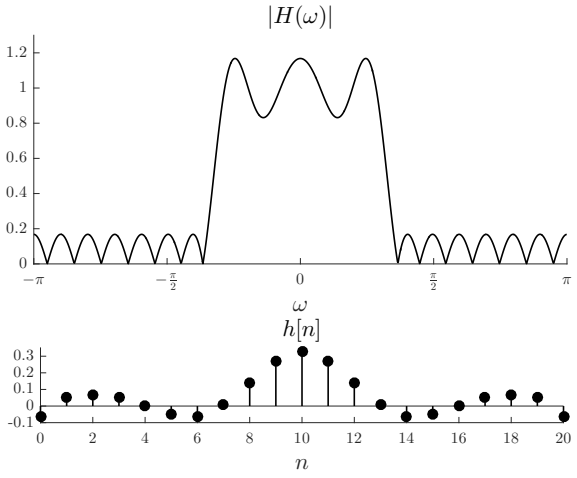
- **Alternation Theorem:** The optimal $A(\omega)$ will touch the error bounds $M/2+2$ times in the interval $0 \le \omega < \pi$

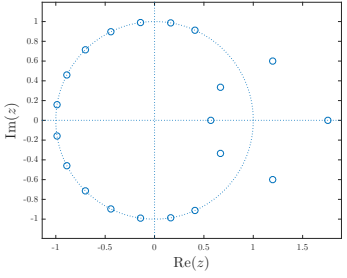
- Parks and McClellan proposed the **Remez Exchange Algorithm** to find the $h[n]$ such that $A(\omega)$ satisfies the alternation theorem

- Matlab command `firpm` and `firpmord` (be careful with the parameters)

Example 1: Optimal FIR Filter Design (1)

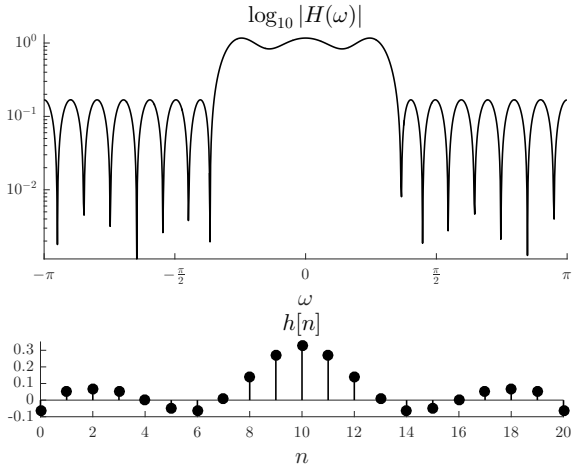
- Optimal low-pass filter of length $M + 1 = 21$ with $\omega_p = 0.30\pi$, $\omega_s = 0.35\pi$
- Note the $M/2 + 2 = 12$ alternations

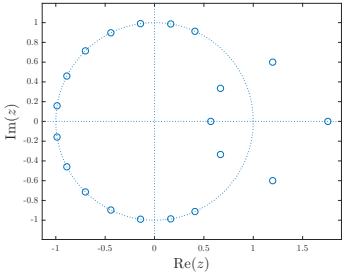




Example 1: Optimal FIR Filter Design (2)

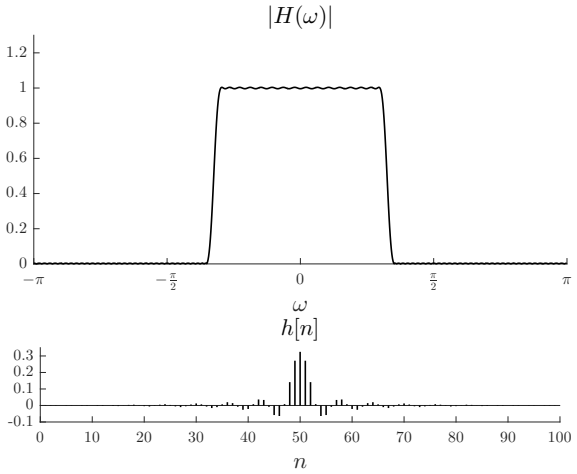
- Optimal low-pass filter of length $M + 1 = 21$ with $\omega_p = 0.30\pi$, $\omega_s = 0.35\pi$
- Note the $M/2 + 2 = 12$ alternations

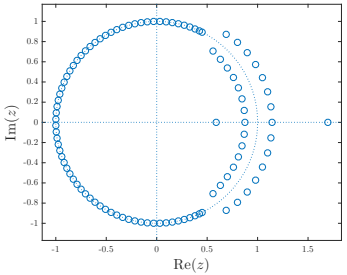




Example 2: Optimal FIR Filter Design (1)

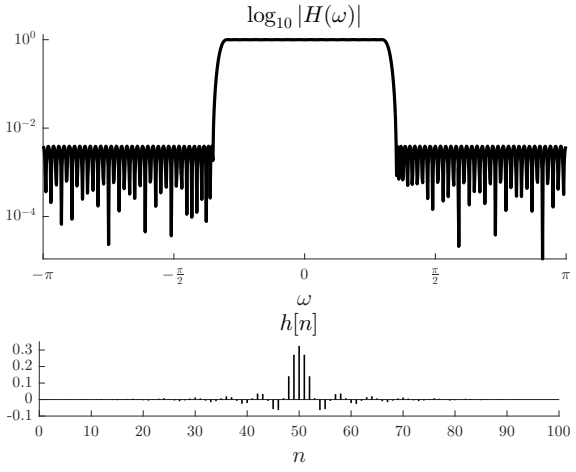
- Optimal low-pass filter of length $M + 1 = 101$ with $\omega_p = 0.30\pi$, $\omega_s = 0.35\pi$
- Note the $M/2 + 2 = 52$ alternations

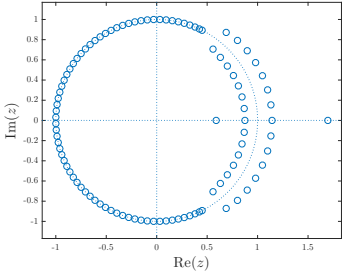




Example 2: Optimal FIR Filter Design (2)

- Optimal low-pass filter of length $M + 1 = 101$ with $\omega_p = 0.30\pi$, $\omega_s = 0.35\pi$
- Note the $M/2 + 2 = 52$ alternations





Matlab Example: Optimal FIR Filter Design

- Process a chirp signal through an optimal low-pass filter with
  - Length $M + 1 = 101$
  - $\omega_p = \pi/3$
  - $\omega_s = \pi/2$
- **Click here** to view a video demonstration.

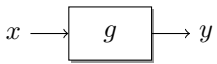
Summary

- FIR filters correspond to a moving average and have **only zeros** (no poles)
- FIR filters are specific to discrete-time; they cannot be built in analog using R, L, C
- Symmetrical FIR filters have (generalized) linear phase, which is impossible with IIR or analog filters
- Design **optimal** FIR filters using the Parks-McClellan algorithm (Remez exchange algorithm)
- FIR filters are always BIBO stable and very numerically stable (to coefficient quantization, etc.)
- Generally require a higher complexity to meet a given spec than an IIR filter, but the benefits can outweigh the computational cost

Inverse Filter and Deconvolution

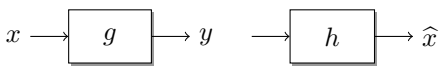
LTI Signal Degradations

- In many important applications, we do not observe the signal of interest x but rather a version y processed by an LTI system with impulse response g



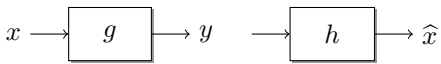
- Examples:
  - Digital subscriber line (DSL) communication (long wires)
  - Echos in audio signals
  - Camera blur due to misfocus or motion (2D)
  - Medical imaging (CT scans), ...

- **Goal:** Ameliorate the degradation by passing y through a second LTI system in the hopes that we can "cancel out" the effect of the first such that $\widehat{x} = x$



LTI Signal Degradations in the z-Transform Domain

- **Goal:** Ameliorate the degradation by passing y through a second LTI system in the hopes that we can "cancel out" the effect of the first such that $\widehat{x} = x$



- Easy to understand using z-transform

$$\widehat{X}(z) \;=\; H(z)\,Y(z) \;=\; H(z)\,G(z)\,X(z)$$

- Therefore, in order to have $\widehat{x} = x$, and thus $\widehat{X}(z) = X(z)$, we need

$$H(z)\,G(z) \;=\; 1 \quad \text{or} \quad H(z) \;=\; \frac{1}{G(z)}$$

- $H(z) = \frac{1}{G(z)}$ is called the **inverse filter**, and this process is called **deconvolution**

Inverse Filter – Poles and Zeros

- If the degradation filter $G(z)$ is a rational function with zeros $\{\zeta_i\}$ and poles $\{p_j\}$

$$G(z) \;=\; z^{N-M}\,\frac{(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

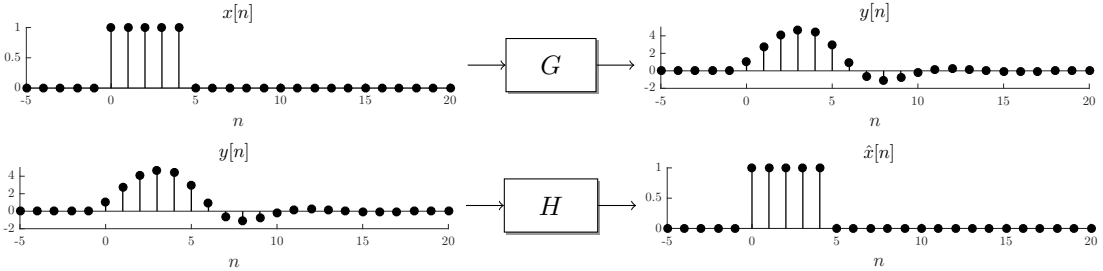
  then the inverse filter $H(z)$ is a rational function with zeros $\{p_j\}$ and poles $\{\zeta_i\}$

$$H(z) \;=\; \frac{1}{G(z)} \;=\; z^{M-N}\,\frac{(z-p_1)(z-p_2)\cdots(z-p_N)}{(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_M)}$$

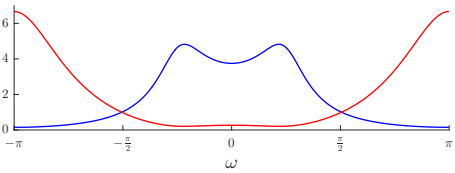
- Assuming that $G(z)$ and $H(z)$ are causal, if any of the zeros of $G(z)$ are outside the unit circle, then $H(z)$ is **not BIBO stable**, which means that the inverse filter does not exist

- When $G(z)$ is causal and all of its zeros are inside the unit circle, we say that it has **minimum phase**; in this case an exact inverse filter $H(z)$ exists

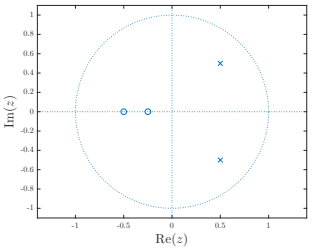
Example: Exact Inverse Filter



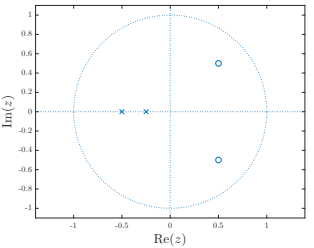
$G(\omega)$ (blue) and $H(\omega) = \frac{1}{G(\omega)}$ (red)



$G(z)$



$H(z)$



Approximate Inverse Filter

- When $G(z)$ is non-minimum phase, an exact inverse filter does not exist, because $\frac{1}{G(z)}$ has one or more poles outside the unit circle

- We can still find an **approximate** inverse filter by **regularizing** $\frac{1}{G(z)}$; for example

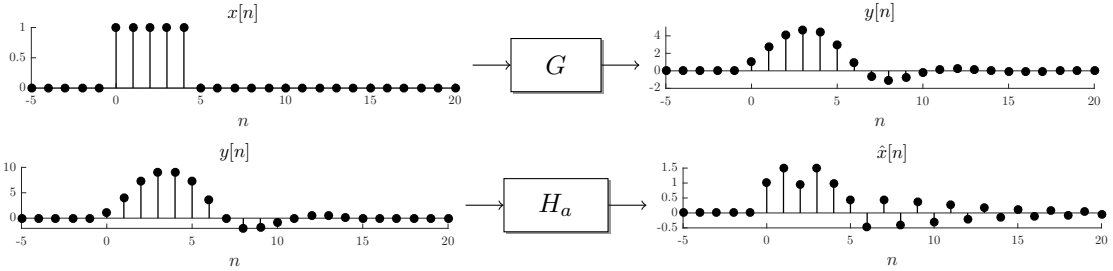
$$H_a(z) \;=\; \frac{1}{G(z)+r}$$

  where r is a constant (technically this is called Tikhonov regularization)

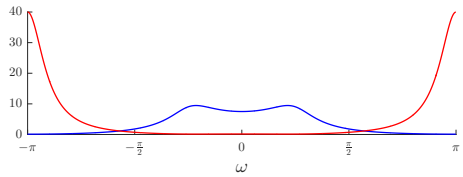
- Typically we try to choose the smallest r such that $H_a(z)$ is BIBO stable

- We no longer have $\widehat{x} = x$, but rather $\widehat{x} \approx x$

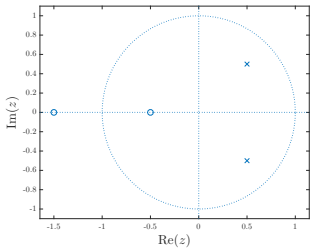
Example: Approximate Inverse Filter



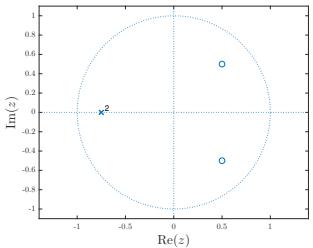
$G(\omega)$ (blue) and $H_a(\omega) = \frac{1}{G(\omega)+\frac{1}{16}}$ (red)



$G(z)$

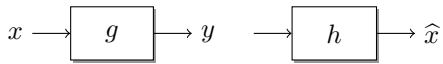


$H_a(z)$



Summary

- **Deconvolution:** Ameliorate the degradation from an LTI system G by passing the degraded signal through a second LTI system H in the hopes that we can "cancel out" the effect of the first such that $\widehat{x} = x$



- **Inverse filter:** Poles/zeros of $G(z)$ become zeros/poles of $H(z)$

$$H(z) = \frac{1}{G(z)}$$

- Best case: When $G(z)$ is causal and all of its zeros are inside the unit circle, we say that it has **minimum phase**; in this case an exact inverse filter $H(z)$ exists
- Puzzler: What do we do when $N \neq M$ in $G(z)$?
- Advanced topics beyond the scope of this course: blind deconvolution, adaptive filters (LMS alg.)

Matched Filter

Inner Product and Cauchy Schwarz Inequality

- Recall the **inner product** (or dot product) between two signals x, y (whether finite- or infinite-length)

$$\langle y, x \rangle \;=\; \sum_n y[n]\, x[n]^*$$

- Recall the **Cauchy-Schwarz Inequality** (CSI)

$$0 \;\leq\; |\langle y, x \rangle| \;\leq\; \|y\|_2 \, \|x\|_2$$

- Interpretation: The inner product $\langle y, x \rangle$ measures the **similarity** of y and x
  - Large value of $|\langle y, x \rangle| \;\Rightarrow\; y$ and x very similar
  - Small value of $|\langle y, x \rangle| \;\Rightarrow\; y$ and x very disimilar

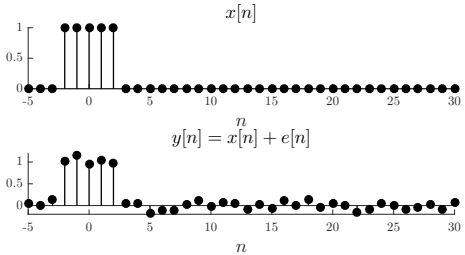
Signal Detection Using Inner Product

- We can determine if a **target signal** x of interest is present within a given signal y simply by computing the inner product and comparing it to a threshold $t > 0$

$$|d| \;=\; |\langle y, x\rangle| \;\begin{cases} \geq & t \quad \text{signal is present} \\ < & t \quad \text{signal is not present} \end{cases}$$

(Aside: In certain useful cases, this is the optimal way to detect a signal)

- Example: Detect the square pulse x in a noisy version $y = x + e$; we calculate $|d| = |\langle y, x\rangle| = 4.92$

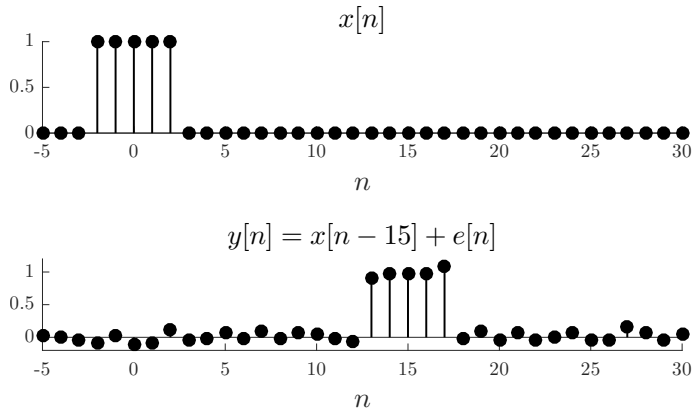


Signal Detection With Unknown Shift

- In many important applications, the target is **time-shifted** by some unknown amount ℓ

$$y[n] \;=\; x[n-\ell] + e[n]$$

- Example: Square pulse with shift $\ell = 15$



Solution: Signal Detection With Unknown Shift

- In many important applications, the target is **time-shifted** by some unknown amount ℓ

$$y[n] \;=\; x[n-\ell] + e[n]$$

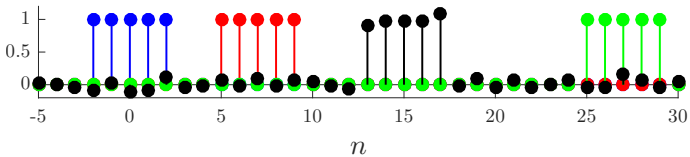
- **Solution:** Compute inner product between y and **shifted target signal** $x[n-m]$ for all $m \in \mathbb{Z}$

$$|d[m]| \;=\; \big|\langle y[n], x[n-m]\rangle\big|$$

- In statistics, $d[m]$ is called the **cross-correlation**; it provides both
  - The detection statistic to compare against the threshold t for each value of shift m
  - An estimate for ℓ (the m the maximizes $d[n]$)

- Example: Square pulse with shift $\ell = 15$ and $m = 0, 7, 27$

$$y[n] = x[n-15] + e[n]$$



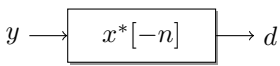
Matched Filter

- Useful interpretation of the cross correlation: Let $\widetilde{x}[n] = x^*[-n]$; then

$$d[m] \;=\; \langle y[n], x[n-m] \rangle \;=\; \sum_n y[n]\, x^*[n-m] \;=\; \sum_n y[n]\, \widetilde{x}[m-n]$$

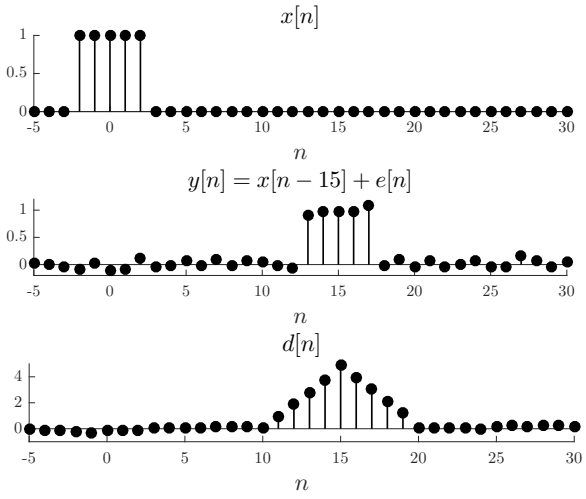
- In words, the cross-correlation $d[m]$ equals the **convolution** of $y[n]$ with the time-reversed and conjugated target signal $\widetilde{x}[n] = x^*[-n]$

- $\widetilde{x}[n] = x^*[-n]$ is the **impulse response** of the **matched filter**



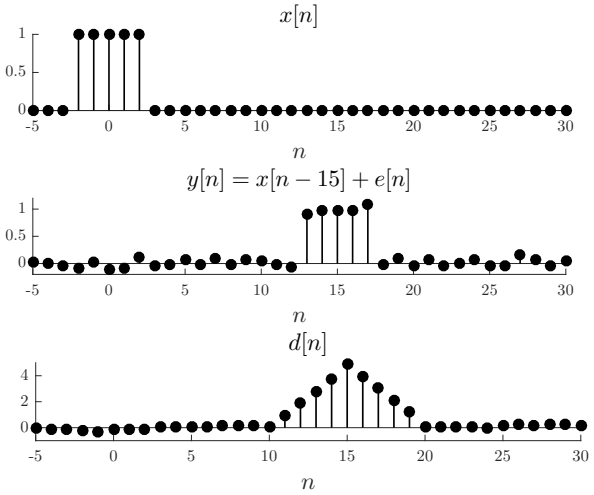
Example: Matched Filter

- Square pulse shifted by $\ell = 15$: $y[n] = x[n-15] + e[n]$



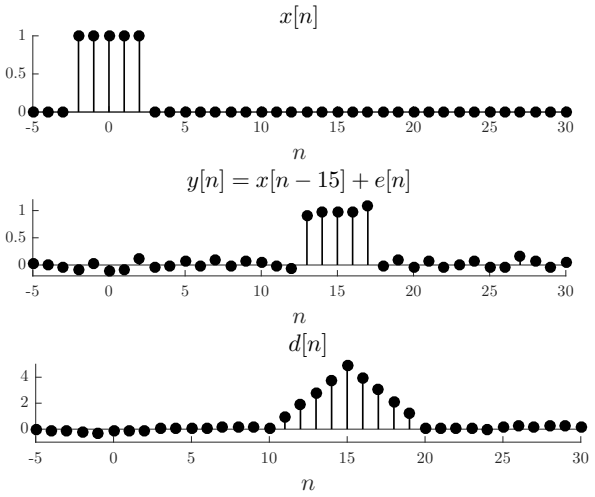
Application: Radar Imaging (1)

- In a radar system, the time delay ℓ is linearly proportional to $2\times$ the distance between the antenna and the target



Application: Radar Imaging (2)

- In a radar system, the time delay ℓ is linearly proportional to $2\times$ the distance between the antenna and the target



Summary

- Inner product and Cauchy Schwarz Inequality provide a natural way to detect a target signal x embedded in another signal y
  - Compare magnitude of inner product to a threshold
- When the target signal is time-shifted by an unknown time-shift ℓ, compute the **cross-correlation**: inner products at all possible time shifts
- Cross-correlation can be interpreted as the convolution of the signal y with a time-reversed and conjugated version of x: the **matched filter**
- Matched filter is ubiquitous in signal processing: radar, sonar, communications, pattern recognition ("Where's Waldo?"), ...