1 Discrete-time Signals

In previous classes we saw that we can create discrete-time signals from continuous time signals. In general, however, a discrete-time signal is any function over the integers. Equivalently we may refer to these signals as sequences, since a sequence is simply a set of numbers indexed by the integers. For example, the discrete-time complex exponential signal is

\[ s(n) = e^{j2\pi fn}. \]

This is one of the most important signals, as we shall see when we learn the discrete-time Fourier transform. Note that in the above equation, \( f \) has no units. For any integer \( m \), if the \( f \) is shifted by \( m \), there is no change in the signal.

\[ s_1(n) = e^{j2\pi(f+m)n}, \]
\[ = e^{j2\pi fn}e^{j2\pi mn}, \]
\[ = e^{j2\pi fn} \text{ since } e^{j2\pi k} = 1 \text{ for any integer } k, \]
\[ = s(n) \]

Any integer shift of \( f \) does not change the signal.

The second-most important discrete-time signal is the unit sample, or Kronecker delta.

This function only takes on one non-zero values. It is defined as

\[ \delta_n = \delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}. \]

If we return to the complex exponential sequence, it looks like a sum of shifted and scaled unit samples. It turns out that any discrete-time signal can be written as

\[ s(n) = \sum_{m=-\infty}^{\infty} s(m)\delta(n - m). \]
In continuous time we had a unit step function; this function also exists in discrete time. The unit-step function is defined as

\[ u(n) = \begin{cases} 
1 & n \geq 0 \\
0 & n < 0 
\end{cases} \]

Note that in continuous the value at \( t = 0 \) was undefined, but in discrete-time we have \( u(0) = 1 \).

Discrete-time signals may also be referred to as sequences. This is more common when the range of the signal is not the set of real numbers. For example, the function could a mapping to a set of bit sequences. In a case like this, we refer to the range of the sequence/signal as the alphabet.

## 2 Discrete-time Fourier Transform

As with analog (continuous-time) signals, we wish to analyze the spectrum of the discrete-time signals. The Fourier transform previously learned was the Continuous-time Fourier Transform (CTFT), so it’s no surprise for the discrete-time signals we have a Discrete-time Fourier Transform (DTFT). The DTFT is defined as

\[ \mathcal{F} \{s(n)\} = S(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} s(n)e^{-j2\pi fn}. \]

This function is continuous in \( f \). The DTFT has several of the same properties as the CTFT:

- the DTFT is linear

- if \( s(n) \) is real-valued, we have conjugate symmetry: \( S(e^{-j2\pi f}) = S(e^{j2\pi f})^* \).

- a shift in the time domain corresponds to multiplication by an exponential in the frequency domain

This list is by no means complete, it is just an example of the several properties that still hold for the Fourier transform in discrete-time.
The most important property is that the DTFT is periodic with period $T = 1$, whether or not $s(n)$ is periodic.

$$S \left( e^{j2\pi f} \right) = \sum_{n = -\infty}^{\infty} s(n) e^{-j2\pi(fn)},$$

$$= \sum_{n = -\infty}^{\infty} s(n) e^{-j2\pi fn} e^{-j2\pi n},$$

$$= \sum_{n = -\infty}^{\infty} s(n) e^{-j2\pi fn},$$

$$= S \left( e^{j2\pi f} \right)$$

This means that we only need to plot the DTFT over one period to know its structure. For any signal $s(n)$, we need only plot $S \left( e^{j2\pi f} \right)$ for $f$ in the interval $[0, 1]$, or any interval of length one.

**Example 1.** Find the DTFT of the signal

$$s(n) = a^n u(n).$$

$$S \left( e^{j2\pi f} \right) = \sum_{n = -\infty}^{\infty} a^n u(n) e^{-j2\pi fn},$$

$$= \sum_{n = 0}^{\infty} a^n e^{-j2\pi fn},$$

$$= \sum_{n = 0}^{\infty} (ae^{-j2\pi f})^n$$

This is a geometric series. The series converges if and only if

$$\left| ae^{-j2\pi f} \right| < 1,$$

which is true as long as $|a| < 1$. The convergent series yields

$$S \left( e^{j2\pi f} \right) = \frac{1}{1 - ae^{-j2\pi f}}.$$

**Example 2.** Find the Fourier transform for the shifted unit sample function

$$s(n) = \delta(n - m).$$

$$S \left( e^{j2\pi f} \right) = \sum_{n = -\infty}^{\infty} \delta(n - m) e^{-j2\pi fn},$$

$$= e^{-j2\pi fm}.$$

We can easily see that the magnitude of the DTFT is one for all values of $f$. Thus, the DTFT is that of an all-pass filter.

We also have a formula for the inverse DTFT which is

$$s(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S \left( e^{j2\pi f} \right) e^{j2\pi fn} df.$$

The limits of the integral need not be $-\frac{1}{2}$ and $\frac{1}{2}$; any limits with a difference of one is find since the DTFT is periodic with a period of one.