1 Review of DFT

Definition 1. Discrete Fourier Transform (DFT): For \( s(n) \) non-zero for \( n = 0, ..., N - 1 \), the DFT is

\[
S(k) = \sum_{n=0}^{N-1} s(n)e^{-j\frac{2\pi kn}{N}} \quad \text{for} \quad k = 0, ..., K - 1.
\]

The above is a DFT of length \( K \). If \( K = N \), we can also write a formula for the inverse DFT,

\[
s(n) = \frac{1}{N} \sum_{k=0}^{N-1} S(k)e^{j\frac{2\pi kn}{N}}.
\]

Example 1. Problem 5.16 Part a. Suppose you have a length \( N \) boxcar sequence. What is the length \( K \) DFT for \( K > N \)? We use the above formula for the DFT.

\[
X(k) = \sum_{n=0}^{N-1} e^{-j\frac{2\pi nk}{N}},
\]

\[
= \frac{1 - e^{-j\frac{2\pi Nk}{K}}}{1 - e^{-j\frac{2\pi k}{K}}} \quad \text{(finite Geometric series)}.
\]

Part b. Let \( K = 4 \). Find the inverse DFT of the product of two DFTs for \( N = 3 \) length boxcars.

\[
X(k) = 1 + e^{-j\frac{2\pi k}{4}} + e^{-j\frac{2\pi 2k}{4}},
\]

\[
Y(k) = \left(1 + e^{-j\frac{2\pi k}{4}} + e^{-j\frac{2\pi 2k}{4}}\right)^2,
\]

\[
= 1 + e^{-j\frac{2\pi k}{4}} + e^{-j\frac{2\pi 2k}{4}} + e^{-j\frac{2\pi 3k}{4}} + e^{-j\frac{2\pi 4k}{4}} + e^{-j\frac{2\pi 5k}{4}} + e^{-j\frac{2\pi 6k}{4}} + e^{-j\frac{2\pi 7k}{4}} + e^{-j\frac{2\pi 8k}{4}},
\]

\[
y(n) = \delta(n) + 2\delta(n - 1) + 3\delta(n - 2) + 2\delta(n - 3).
\]
Part c. Suppose our filter is a length three boxcar sequence, i.e. it has the following length four DFT:

\[ H(k) = 1 + e^{-\frac{j2\pi k}{4}} + e^{-\frac{j2\pi 2k}{4}}. \]

Is the result in part (b) consistent with the actual output of the boxcar filter? If our FIR filter has length 3 and our input has length three, we know our output will have length \( 3 + 3 - 1 = 5 \). This is inconsistent with the answer in part (b). We should’ve taken length five DFTs instead of length four.

## 2 Communication

Recall our fundamental model for communication.

![Diagram of Communication System](image)

**Figure 2: Communication System.**

First, let’s figure out what the model means. What is the role of the transmitter? The transmitter is responsible for encoding the information signal \( m(t) \), modulating it, and creating the electromagnetic wave that will be transmitted. Whatever the transmitter does to the signal, the receiver should be able to un-do.

In general, \( r(t) \neq x(t) \), which is why we analyze such a system. Our basic model for the received signal is

\[ r(t) = \alpha x(t) + N_t, \]

where \( \alpha \) represents an attenuation factor of the channel and \( N_t \) is additive noise. We assume \( |\alpha| < 1 \), meaning that power is lost during transmission. There is no exact function to model how air attenuates a signal, but we have models that work, and there are methods to combat this attenuation. However, we never know the noise, which will typically modelled as a Normal (Gaussian) random process.

In wireless communication, the received power of the signal is related to the distance over which it travels:

\[ \text{Received signal power} \propto \frac{1}{d^\beta}, \]

where \( \beta \in [2, 4] \). The \( \beta \) factor is referred to as the path loss coefficient. The power of a signal diminishes exponentially with respect to distance.

What is the power of a signal?

**Definition 2.** Power Spectrum:

\[ P_s(f) = |S(f)|^2, \]

where \( S(f) \) is the CTFT of the signal \( s(t) \).

By Parseval’s Theorem, the total power of a signal is

\[ \int_{-\infty}^{\infty} |S(f)|^2 df. \]

The power of a signal contained in the band \([f_1, f_2]\) is

\[ 2 \int_{f_1}^{f_2} |S(f)|^2 df. \]
The factor of two comes from the fact that magnitude of a signal’s spectrum is an even function; the negative frequency components mirror the positive frequency components.

For white noise, the power spectrum is flat, meaning that $P_N(f) = \frac{N_0}{2}$, i.e., its power spectrum is constant for all $f$. Thus, the noise power within the band $[f_1, f_2]$ is

$$2 \int_{f_1}^{f_2} \frac{N_0}{2} = N_0(f_2 - f_1).$$

We define the signal-to-noise ratio (SNR) to be the ratio of the signal power over the noise power.

For a LTI system, the spectrum of the output is the product of the system’s transfer function and the spectrum of the input. Thus,

$$Y(f) = H(f)S(f),$$

$$\Rightarrow |Y(f)|^2 = |H(f)|^2|S(f)|^2,$$

$$\Rightarrow P_y(f) = |H(f)|^2P_s(f).$$

2.1 Modulation

For amplitude modulation (AM), the general form of our transmitted signal is

$$x(t) = A_c(1 + m(t))\cos(2\pi f_c t).$$

The gain, $A_c$, is the carrier amplitude and the frequency, $f_c$, is the carrier frequency. Our message is $m(t)$ and we assume that $|m(t)| < 1$.

Alternatively, we can modulate our signal with sin or exp. In general, to demodulate the signal at the receiver, we do some sequence of filtering the received signal and multiplying it with a sinusoid (depends on which sinusoid you modulated with).