Periodicity, Real Fourier Series, and Fourier Transforms

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1 Periodicity and Fourier Series

The *period* of the a function is the smallest value $T \in \mathbb{R}$ such that $\forall t \in \mathbb{R}$ and any $k \in \mathbb{Z}$,

$$s(t) = s(t + kT).$$

The *fundamental frequency* of a signal is

$$f_0 = \frac{1}{T}.$$

The *harmonics* are the terms are integer multiples of the the fundamental frequency, i.e. the k^{th} harmonic is at frequency $\frac{k}{T}$.

Example 1. Find the period of a basic sine wave. Suppose $s(t) = sin(2\pi ft)$. We want to find T such that

$$\begin{aligned} s(t) &= s(t-T), \\ \Rightarrow sin(2\pi ft) &= sin(2\pi f(t-T)), \\ &= sin(2\pi ft - 2\pi fT). \end{aligned}$$

In order for the last inequality to be true, we require $fT \in \mathbb{Z}$. The smallest value of T that meets this requirement is

$$T = \frac{1}{f},$$

as we should expect.

Example 2. Find the period and Fourier series of

$$s(t) = sin(2\pi 440t) + sin(2\pi 550t) + sin(2\pi 660t).$$

This signal represents the "major triad" consisting of notes A, E, and C#. To find the period, we used the same trick as before.

$$\begin{aligned} s(t-T) &= sin(2\pi 440(t-T)) + sin(2\pi 550(t-T)) + sin(2\pi 660(t-T)), \\ &= sin(2\pi 440t - 2\pi 440T) + sin(2\pi 550t - 2\pi 550T) + sin(2\pi 660t - 2\pi 660T). \end{aligned}$$

We want the smallest T such that $440T, 550T$, and $660T$, are all integers. Equivalently, we want the largest $f_0 = \frac{1}{T}$ such that

$$\frac{440}{T}, \frac{550}{T}, \frac{660}{T} \in \mathbb{Z}.$$

This f_0 is the *greatest common divisor (GCD)* of the three numbers. (Note: there is a GCD function in Matlab.) It should be clear here that $f_0 = 110Hz$, so

$$T = \frac{1}{110}.$$

What is the real Fourier series for this signal? First of all, we know it is a periodic odd signal, so $a_k = 0$ for all k. We re-write the signal:

$$s(t) = sin\left(\frac{2\pi 4t}{\frac{1}{110}}\right) + sin\left(\frac{2\pi 5t}{\frac{1}{110}}\right) + sin\left(\frac{2\pi 6t}{\frac{1}{110}}\right).$$

From the above we know

$$b_k = \begin{cases} 1 & k = 4, 5, 6 \\ 0 & \text{o/w} \end{cases}.$$

Example 3. Find the period and Fourier series of

$$s(t) = 10sin(2\pi t) + (10 + 2j)cos(2\pi t).$$

First, we find the period:

$$\begin{aligned} s(t-T) &= 10sin(2\pi(t-T)) + (10+2j)cos(2\pi(t-T)), \\ &= 10sin(2\pi t - 2\pi T) + (10+2j)cos(2\pi t - 2\pi T). \end{aligned}$$

Since we want $2\pi T$ to be an integer multiple of 2π, we have that $T = 1$. Note that since the signal is complex-valued, we cannot find the real Fourier series. To find the values of c_k, we use Euler's formula:

$$\begin{aligned} s(t) &= \frac{10}{2j}e^{j2\pi t} - \frac{10}{2j}e^{-j2\pi t} + \frac{10+2j}{2}e^{j2\pi t} + \frac{10+2j}{2}e^{-j2\pi t}, \\ &= \frac{8+10j}{2j}e^{j2\pi t} + \frac{10j-12}{2j}e^{-j2\pi t}, \\ &= \frac{4+5j}{j}e^{j2\pi t} + \frac{5j-6}{j}e^{-j2\pi t}, \\ &= (5-4j)e^{j2\pi t} + (5+6j)e^{-j2\pi t}. \end{aligned}$$

Thus,

$$c_k = \begin{cases} 5+6j & k = -1 \\ 5-4j & k = 1 \\ 0 & \text{o/w} \end{cases}.$$

Example 4. Find the period and Fourier series of

$$s(t) = sin(2\pi t) + sin(\sqrt{2}\pi t).$$

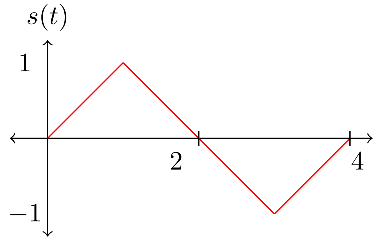
To find the period, we examine $s(t-T)$.

$$\begin{aligned} s(t-T) &= sin(2\pi(t-T)) + sin(\sqrt{2}\pi(t-T)), \\ &= sin(2\pi t - 2\pi T) + sin(\sqrt{2}\pi t - \sqrt{2}\pi T). \end{aligned}$$

We need

$$2\pi T = 2\pi k_1, k_1 \in \mathbb{Z},$$
$$\sqrt{2}\pi T = 2\pi k_2 \in \mathbb{Z}.$$

This implies that $T = \frac{1}{\sqrt{2}}k_2$ and $T = k_1$, which is not possible since T cannot be an integer in both cases. Thus, $T = 0$. The signal is not periodic and therefore does not have a Fourier series.



Example 5. Consider the triangle wave. Find the Fourier series.

Since the function is odd, we know that $a_k = 0$ for all k. Thus, we only need to find the b_k coefficients.

$$
\begin{aligned}
b_k &= \frac{2}{4}\int_0^4 s(t)sin(\frac{2\pi kt}{4})dt, \\
&= \frac{1}{2}\left(\int_0^1 tsin(\frac{\pi kt}{2})dt + \int_1^3 (2-t)sin(\frac{\pi kt}{2})dt + \int_3^4 (t-4)sin(\frac{\pi kt}{2})dt\right), \\
&= \frac{1}{2}\left(\frac{-2t}{\pi k}cos(\frac{\pi kt}{2})|_0^1 + \int_0^1 \frac{2}{\pi k}cos(\frac{\pi kt}{2})dt + \frac{-2(2-t)}{\pi k}cos(\frac{\pi kt}{2})|_1^3 - \int_1^3 \frac{2}{\pi k}cos(\frac{\pi kt}{2})dt + \frac{-2(t-4)}{\pi k}|_3^4 + \int_3^4 \frac{2}{\pi k}cos(\frac{\pi kt}{2})dt\right), \\
&= \frac{1}{2}\left(\frac{-2}{\pi k}cos(\frac{\pi k}{2}) + \frac{4}{(\pi k)^2}sin(\frac{\pi kt}{2})|_0^1 + \frac{2}{\pi k}cos(\frac{3\pi k}{2}) + \frac{2}{\pi k}cos(\frac{\pi k}{2}) - \frac{4}{(\pi k)^2}sin(\frac{\pi kt}{2})|_1^3 + \frac{-2}{\pi k}cos(\frac{3\pi k}{2}) + \frac{4}{(\pi k)^2}sin(\frac{\pi kt}{2})|_3^4\right), \\
&= \frac{1}{2}\left(\frac{8}{(\pi k)^2}sin(\frac{\pi k}{2}) - \frac{8}{(\pi k)^2}sin(\frac{3\pi k}{2})\right), \\
&= \begin{cases} \frac{8}{(\pi k)^2} & k = 1, 5, 9, ... \\ \frac{-8}{(\pi k)^2} & k = 3, 7, 11, ... \\ 0 & \text{o/w} \end{cases} .
\end{aligned}
$$

2 Fourier Transforms

For a any continuous signal, $s(t)$, we define its *Fourier transform* to be

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt.$$

The *inverse Fourier transform* returns the signal from its transform:

$$s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}dt.$$

Note that for the Fourier transform we integrate over t (time), so that our expression becomes a function of f (frequency).

Example 6. Find the Fourier transform of

$$s(t) = e^{-at}u(t),$$

for some real number a.

$$\begin{aligned}
S(f) &= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j2\pi f t} dt, \\
&= \int_{0}^{\infty} e^{-(a+j2\pi f)t} dt, \\
&= \frac{-1}{a+j2\pi f} e^{-(a+j2\pi f)t}\Big|_0^{\infty}, \\
&= \frac{1}{a+j2\pi f}.
\end{aligned}$$

Example 7. Find the Fourier transform of the pulse of length Δ centered about zero.

$$\begin{aligned}
P_\Delta(f) &= \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^{-j2\pi f t} dt, \\
&= \frac{-1}{j2\pi f} e^{-j2\pi f t}\Big|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}, \\
&= \frac{-1}{j2\pi f}\left(e^{-j2\pi f \frac{\Delta}{2}} - e^{j2\pi f \frac{\Delta}{2}}\right), \\
&= \frac{1}{\pi f} sin(\pi f \Delta), \\
&= \Delta sinc(\pi f \Delta).
\end{aligned}$$

It turns out that if the time-domain signal is a *sinc* function, the Fourier transform is a pulse.

$$s(t) = 2W sinc(2\pi W t) \leftrightarrow S(f) = \begin{cases} 1 & |f| < W \\ 0 & |f| > W \end{cases}$$

There are actually many nice properties of Fourier transforms. Below is a list of some of them:

- Linearity:

$$a_1 s_1(t) + a_2 s_2(t) \leftrightarrow a_1 S_1(f) + a_2 S_2(f).$$

- Conjugate symmetry: if $s(t) \in \mathbb{R}$, then $S(f) = S(-f)^*$. This implies that $|S(f)|$ is an even function and $\angle S(f)$ is an odd function (similar to the property for Fourier series of a real periodic signal).

- Even symmetry: if $s(t)$ is an even function, then $S(f)$ is an even function.

- Odd symmetry: if $s(t)$ is an odd function, then $S(f)$ is an odd function.

- Scaling: for $a \in \mathbb{R}$,

$$s(at) \leftrightarrow \frac{1}{|a|} S\left(\frac{f}{a}\right).$$

- Time delay: a delay of τ in the time domain corresponds to multiplication by an exponential in the frequency domain.

$$s(t-\tau) \leftrightarrow e^{-j2\pi f \tau} S(f)$$

- Complex modulation: multiplication by an exponential in the time domain corresponds to a frequency shift in the frequency domain.

$$e^{j2\pi f_0 t} s(t) \leftrightarrow S(f - f_0)$$