1 Fourier Transform

Recall the formulas for the Fourier transform:

\[ \mathcal{F}\{s(t)\} = S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt, \]

\[ \mathcal{F}^{-1}\{S(f)\} = s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df. \]

Suppose \( s(t) \) is periodic. Then we can write it using the Fourier series,

\[ s(t) = \sum_{l=-\infty}^{\infty} c_l e^{j2\pi lt}. \]

We can compute the Fourier transform of the signal using its Fourier series representation.

\[ S(f) = \int_{-\infty}^{\infty} \left( \sum_{l=-\infty}^{\infty} c_l e^{j2\pi lt} \right) e^{-j2\pi ft} dt, \]

\[ = \sum_{l=-\infty}^{\infty} c_l \int_{-\infty}^{\infty} e^{j2\pi lt} e^{-j2\pi ft} dt, \]

\[ = \sum_{l=-\infty}^{\infty} c_l \delta\left( f - \frac{l}{T} \right). \]

The function \( \delta(t) \) is the Dirac delta function:

\[ \delta(t) = \begin{cases} 
1 & t = 0 \\
0 & t \neq 0. 
\end{cases} \]

This means that in order to find the Fourier transform of a periodic signal, we only need to find the Fourier series coefficients.

Example 1. Find the Fourier transform of

\[ s(t) = \cos(2\pi f_0 t). \]

We can re-write the signal using Euler’s formula:

\[ s(t) = \frac{1}{2} e^{j2\pi f_0 t} + \frac{1}{2} e^{-j2\pi f_0 t}. \]
Thus, the Fourier series coefficients are

\[ a_k = \begin{cases} 
\frac{1}{2} & k = -1, 1, \\
0 & \text{o/w}
\end{cases} \]

and so the Fourier transform is

\[
S(f) = a_{-1} \delta \left( f + \frac{1}{T} \right) + a_1 \delta \left( f - \frac{1}{T} \right),
\]

\[
= \frac{1}{2} \delta (f + f_0) + \frac{1}{2} \delta (f - f_0).
\]

In general, for non-periodic signals, the Fourier transform has many nice properties. I recommend looking at CTFT tables online or in the course book. Two nice properties to highlight are the operations of differentiation and integration in the time domain. Consider a signal \( s(t) \) and take the derivative.

\[
y(t) = \frac{d}{dt} s(t),
\]

\[
= \frac{d}{dt} \int_{-\infty}^{\infty} S(f)e^{j2\pi ft} df,
\]

\[
= \int_{-\infty}^{\infty} S(f) \frac{d}{dt} \{e^{j2\pi ft}\} df,
\]

\[
= \int_{-\infty}^{\infty} S(f)j2\pi f e^{j2\pi ft} df,
\]

\[
= \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft} df.
\]

Thus,

\[
\frac{d}{dt} s(t) \leftrightarrow j2\pi f S(f).
\]

Differentiating in the time domain corresponds to multiplying by \( j2\pi f \) in the frequency domain. Similarly, we can show that integrating in the time domain corresponds to dividing by \( j2\pi f \) in the frequency domain (if \( S(0) = 0 \)).

\[
\int_{-\infty}^{t} s(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f} S(f)
\]

2 Sampling

We can create discrete-time signals by sampling continuous-time signals at regular intervals of length \( T_s \). If we multiply a continuous-time signal \( s(t) \) with a train of Dirac delta functions, we have a sampled signal:

\[
s(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} s(t) \delta(t - nT_s),
\]

\[
= \sum_{n=-\infty}^{\infty} s(nT_s) \delta(t - nT_s).
\]

The sampling period, or interval, is \( T_s \), and the sampling frequency, or rate, is \( f_s = \frac{1}{T_s} \). If \( s(t) \) is band-limited, we can prevent aliasing (overlap in the frequency domain) by selecting \( T_s \) such that

\[ T_s < \frac{1}{2W}. \]
where $W$ is the bandwidth of the signal. This is the as the **Nyquist-Shannon Sampling theorem**. We refer to $f = \frac{1}{2T_s}$ as the **Nyquist frequency** since it is the highest frequency at which a signal can contain energy and remain compatible with the sampling theorem.

If we wish to filter a discrete-time signal that originates from a continuous-time signal, does it matter the order in which we perform the operations, i.e. if we sample and then filter versus filter and then sample? Consider the two systems shown below:

\[ s(t) \xrightarrow{\text{Sampler}} s(n) \xrightarrow{H(f)} y_1(n) \]

**Figure 1:** System 1: Sampling and then filtering.

\[ s(t) \xrightarrow{H(f)} x(t) \xrightarrow{\text{Sampler}} y_2(n) \]

**Figure 2:** System 2: Filtering and then sampling.

Does $y_1(n) = y_2(n)$ for these systems? In class we showed that

\[ \mathcal{F}\{s(n)\} = \sum a_k S\left(f - \frac{k}{T_s}\right). \]

We will use this result in order to show that, in fact, the two signals are **not** equal. In the first system, the Fourier transform for $s(n)$, the output of the sampler, is exactly the formula we have above. If we put this signal through a LTI filter, the Fourier transform of the output is

\[ Y_1(f) = \mathcal{F}\{s(n)\}H(f), \]

\[ = \sum a_k S\left(f - \frac{k}{T_s}\right)H(f). \]

For the second system, we know that

\[ X(f) = S(f)H(f), \]

and then we can use the same formula above for the spectrum of the sampled signal:

\[ Y_2(f) = \mathcal{F}\{y_2(n)\}, \]

\[ = \sum b_k X(f - \frac{k}{T}), \]

\[ = \sum b_k S\left(f - \frac{k}{T}\right)H\left(f - \frac{k}{T}\right). \]

We can see in general the two formulas are not equal. Let’s consider the above problem in pictures.

Sampling the signal creates multiples copies of the spectrum of the signal centered at different frequencies.

If we low-pass filter this sampled signal using a filter with passband size $WHz$, as in the first system, then we will get the original signal back and the spectrum $Y_1(f)$ is the same as $S(f)$. If we use the LPF first on $s(t)$, the spectrum is unchanged since it falls within the passband. If we then sample, as in the second system, the spectrum of the output, $Y_2(f)$, is the spectrum of $s(nT)$. Thus, the two signals are different since their frequency content is different.
Example 2. What should the sampling period be for the sinc function, 

\[ s(t) = \text{sinc}(\pi t) \]?

Recall that the sinc function is defined as

\[ \text{sinc}(\pi t) = \frac{\sin(\pi t)}{\pi t}. \]

We know that

\[ \frac{\sin(2\pi W t)}{\pi t} \rightarrow S(f) = \begin{cases} 1 & -W < f < W \\ 0 & \text{o/w} \end{cases}. \]

Thus, our signal has bandwidth \( W = \frac{1}{2} \). By the sampling theorem, we require

\[ T < \frac{1}{2 \left( \frac{1}{2} \right)} = 1. \]

Any sampling interval less than one will suffice (equivalently, any sampling frequency greater than one). If we select \( T = \frac{1}{2} \), then

\[
    s(n) = \frac{\sin \left( \frac{\pi}{2} n \right)}{\frac{\pi}{2} n},
\]

\[
    = \begin{cases} 
        1 & n = 0 \\
        0 & n \text{ even} \\
        \frac{2}{\pi^2} & n = 1, 5, 9, \ldots \\
        \frac{4}{\pi^2} & n = 3, 7, 11, \ldots 
    \end{cases}.
\]