On the Outage Probability of a Class of Signaling Schemes for Multiple Antennas

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Abstract

In this work, we investigate large SNR behavior of the outage probability for a class of signaling schemes designed for flat fading Rayleigh multiple antenna systems. The analysis is performed for systems with a short-term power constraint, thus time power control is not applicable. Our main result is the characterization of a large class of signaling schemes, all of which are guaranteed to have the identical rate of decay of outage probability with SNR, given by the maximum spatial diversity available in the system. We show that several important multiple antenna signaling schemes, like space time coding and beamforming, are included in the proposed class of signaling schemes. Such a classification of schemes, based on similar error performance, helps in addressing practical issues like tradeoffs in feedback resources and decoding complexity required by each scheme. Finally, the techniques developed in this paper can also be used to derive the diversity gain of an arbitrary signaling scheme.

1 Introduction

Multiple antenna systems have been the focus of recent research owing to their tremendous potential to achieve significantly higher transmission rates [1, 2]. Different coding strategies have been proposed to tap the gains promised by the multiple antenna systems, see *e.g.* [3–5]. The coding strategies largely depend on the nature and the extent of the channel state information available at the transmitter and the receiver in addition to channel statistics.

The concept of outage probability is appropriate for delay-constrained transmission over quasi-static fading channels [6]. Outage probability serves as a useful lower bound to frame error rate for all coded systems. In this work, we analyze different spatial power allocation and beamforming methods, derived with varying amount of channel information at the transmitter and receiver. We characterize a class of schemes, each member of which exhibits an identical asymptotic (in SNR) outage probability decay, given by maximum spatial diversity available in the system. In particular, the outage probability decays as $\left(\frac{1}{SNR}\right)^r$, where r is the product of the number of transmit and receive antennas.

The utility of the proposed class characterization can be summarized as follows. First, we provide a sufficient condition characterizing schemes which guarantee full spatial diversity. The proof technique can be easily extended to characterize all schemes with a certain diversity order, or can be used to find the diversity order of an arbitrary signaling scheme. Second, the results can be seen as a step towards understanding the tradeoff between extent of channel information, outage performance and complexity of the schemes. With a metric on the resource premium [7] required to obtain channel information at the receiver (*e.g.* via training) and the transmitter (*e.g.* using feedback), and a metric on complexity of resulting optimal methods, the system designers can choose among schemes with the same diversity advantage; the results on tradeoffs will be presented elsewhere.

The class of schemes characterized in this paper includes both space time coding and spatial water filling, representing two extremes of channel state information at the transmitter. Space time codes do not require any channel information at the transmitter, thereby saving feedback resources, but have a decoding

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complexity exponential in the number of transmit antennas [3]. On the other hand, spatial water filling requires complete channel information at the transmitter with decoding complexity linear in the number of transmit antennas. Several important signaling schemes are also members of the proposed class, which are designed with different amount of transmitter information and require varying amount of computational resources. It should be noted that the behavior of the ergodic capacity of the member schemes is markedly different for large SNR, even without power control over time[1]. The absence of power control is justified for the case of short term power constraint, where power saved from one frame cannot be used for transmission in a future frame.

The paper is organized as follows. The problem is set up in Section 2 and the notation is introduced. We define a class of signaling schemes in Section 3 for which the asymptotic nature of the outage probability is established in Section 4. In Section 5 we give some examples of well known schemes for which the results of this work hold. We conclude in Section 6.

2 Problem Setting

Consider a wireless communication system with m transmit and n receive antennas. Let H denote the $n \times m$ channel matrix with *i.i.d.* complex entries. Each element of H is assumed to be circular symmetric complex Gaussian with zero mean and unit variance. Let $X \in \mathbb{C}^m$ be the codeword transmitted from the m antennas while $Y \in \mathbb{C}^n$ is the corresponding vector received. We assume that the transmitter is constrained in its total power to P, i.e., $E\left[X^\dagger X\right] \leq P$, where $X^\dagger$ denotes the Hermitian transpose of the matrix X. Assuming flat fading, we have

$$Y = HX + N, \tag{1}$$

where $N \in \mathbb{C}^n$ is the noise vector with *i.i.d.* circular symmetric complex Gaussian entries with zero mean and unit variance.

We assume a non-ergodic model in time for the fading process. That is, the realization of H is chosen at random at the beginning of each frame and stays fixed for the duration of the frame. Slow fading channel is a good example for this channel model. For non-ergodic channel model, it is known that the Shannon capacity of the channel given in (1) is zero [1]; in such cases, outage probability is more meaningful. Outage probability has been discussed in detail in [6].

For a given rate R and power P, the outage probability $P_{out}(R,P)$ is such that for any rate less than R and any δ there exists a code satisfying the power constraint P for which the error probability is less than δ for all but a set of H whose total probability is less than $P_{out}(R,P)$ [1]. We have,

$$P_{out}(R,P) = \inf_{\{Q:Q\geq 0,\text{trace}(Q)\leq P\}} \mathcal{P}\left[\log\det\left(I_n + HQH^\dagger\right) < R\right] \tag{2}$$

where $Q = E[XX^\dagger]$, the correlation matrix of the input codeword.

In the subsequent sections, we define a class of *signaling schemes* and study their outage probability performance. All schemes in the defined class exhibit full spatial diversity.

3 Class of Signaling Schemes Characterized by Bounded Norm

In the sequel, a *signaling scheme* is understood as a pair of transmit and receive methods. The transmit part of a signaling scheme allocates power and phases to the transmit antennas, and the receiver performs maximum likelihood processing to recover the transmitted codeword.

Space time coding is an example of a signaling scheme, where the available power is equally distributed among all the transmitting antennas and the power from all the receiving antennas is collected and combined using maximum likelihood decoding. Other examples include signaling schemes like selection diversity

[1]For the cases where the transmitter has channel state information, time water-filling is commonly used to improve outage performance [8] derived with a long-term power constraint.

where only one of the transmit antennas (the "best" antenna) may be used for transmission or only one of the antennas at the receiver (again the "best" antenna) for reception.

Analogous to (2), we can define the outage probability for a given *signaling scheme* A as

$$P_{out}^{A}(R,P) = \mathcal{P}\left[I_A(X;Y|H) < R\right], \tag{3}$$

where $I_A(X;Y|H)$ is the maximum mutual information achievable by scheme A.

Definition 1 *Let $\mathcal{S}$ be the class of all signaling schemes whose maximum mutual information is bounded in the following sense. For any given scheme $A \in \mathcal{S}$ and any channel $H \in \mathbb{C}^{n\times m}$, there exists a norm $\|\cdot\|_A$ defined on $\mathbb{C}^{n\times m}$ and positive constants k_1, k_2, n_1, n_2 independent of H, such that the maximum mutual information achievable by the scheme A over channel H, given by $I_A(X;Y|H)$, is bounded above and below by*

$$n_1 \log\left(1 + k_1 P \|H\|_A^2\right) \leq I_A(X;Y|H) \leq n_2 \log\left(1 + k_2 P \|H\|_A^2\right). \tag{4}$$

Note that the norm $\|\cdot\|_A$ should be a valid norm[2] on the whole space, $\mathbb{C}^{n\times m}$. We will show that the class $\mathcal{S}$ is fairly general in nature and it contains most of the "good" signaling schemes. An example of a "poor" scheme, which does not belong to this class, is the one which always turns off one or more transmit antennas.

4 Asymptotic Outage Probability of the Signaling Schemes in $\mathcal{S}$

In this section, we show that outage probability for all the schemes which belong to $\mathcal{S}$ over a Rayleigh fading channel decays as $(SNR)^{-r}$ where $r = mn$ is the spatial diversity available in the system. We first state two results which will be used in the characterization of the outage probability of the schemes in $\mathcal{S}$.

Lemma 2 *[9] Any two norms on a finite dimensional space are equivalent*[3].

From Lemma 2, it follows that if $\|.\|_a$ and $\|.\|_b$ are any two norms on a finite dimensional space E, then for any $x \in E$, there exist positive constants M_1 and M_2, independent of x, such that

$$M_1\|x\|_a \leq \|x\|_b \leq M_2\|x\|_a.$$

Note that the set of complex $n \times m$ matrices, $\mathbb{C}^{n\times m}$, is a finite dimensional vector space and hence Lemma 2 holds for norms defined on $\mathbb{C}^{n\times m}$. In the following lemma, we denote the Frobenius norm by $\|.\|_F$, where $\|H\|_F^2 = \mathrm{trace}(HH^\dagger)$.

Lemma 3 *[10] Let $I_F(X;Y|H) = \log(1+kP\|H\|_F^2)$ where k is any positive constant while P and H are as defined earlier. Also, let $P_{out}^F(R,P) = \mathcal{P}\left[I_F(X;Y|H) < R\right]$. Then an upper bound on $P_{out}^F(R,P)$ is given by*

$$P_{out}^F(R,P) \leq \min_{1\leq i\leq mn} \frac{(2^R-1)^i}{i!}(kP)^{-i} \tag{5}$$

We will use the following large P behavior of $P_{out}^F(R,P)$, proved in [10]. As P increases, $P_{out}^F(R,P) \leq \frac{(2^R-1)^{mn}}{(mn)!}(kP)^{-mn}$, a bound which is asymptotically tight, *i.e.*, $P_{out}^F(R,P) \sim K_F\left(\frac{1}{P}\right)^{mn}$, where K_F is independent of P.

Theorem 4 *Let a channel $H \in \mathbb{C}^{n\times m}$ be given such that the components of H are i.i.d. circular symmetric Gaussian random variables. Then, the probability of outage for any signaling scheme $A \in \mathcal{S}$ when used on the channel H decays as $\left(\frac{1}{P}\right)^{mn}$ for large P, where m is the number of transmit antennas and n is the number of receive antennas.*

[2]Any functional on the vector space which is positive, homogeneous and satisfies the triangle inequality is a valid norm.

[3]Two norms are said to be *equivalent* if they define the same topology.

Proof : Let $A \in \mathcal{S}$ be any signaling scheme. By the definition of $\mathcal{S}$, we have a norm $\|.\|_A$ and positive constants k_1, k_2, n_1, n_2 such that

$$n_1 \log\left(1 + k_1 P \|H\|_A^2\right) \leq I_A(X;Y|H) \leq n_2 \log\left(1 + k_2 P \|H\|_A^2\right).$$

Now using Lemma 2, we have two positive constants l_1 and l_2 such that

$$l_1 \|H\|_F^2 \leq \|H\|_A^2 \leq l_2 \|H\|_F^2. \tag{6}$$

Combining the above two chains of inequalities, we have

$$n_1 \log\left(1 + M_1 P \|H\|_F^2\right) \leq I_A(X;Y|H) \leq n_2 \log\left(1 + M_2 P \|H\|_F^2\right), \tag{7}$$

where $M_1 = l_1 k_1$ and $M_2 = l_2 k_2$ are positive constants. It immediately follows that

$$\mathcal{P}\left[n_2 \log\left(1 + M_2 P \|H\|_F^2\right) < R\right] \leq \mathcal{P}\left[I_A(X;Y|H) < R\right] \leq \mathcal{P}\left[n_1 \log\left(1 + M_1 P \|H\|_F^2\right) < R\right]. \tag{8}$$

From Lemma 3 we know that, for large P, the upper bound and the lower bound on $\mathcal{P}\left[I_A(X;Y|H) < R\right]$ given above, decay as $\left(\frac{1}{P}\right)^{mn}$. Hence, $P_{out}^A(R,P) = \mathcal{P}\left[I_A(X;Y|H) < R\right]$ also decays as $\left(\frac{1}{P}\right)^{mn}$ for large P. ■

Hence, the outage probability for every scheme in $\mathcal{S}$ is guaranteed to decay as $\left(\frac{1}{P}\right)^{mn}$ for large P, when the channel exhibits Rayleigh fading characteristics.

5 Example Members of $\mathcal{S}$

In this section, we give examples of signaling schemes which belong to $\mathcal{S}$ so that their outage probability will be characterized by Theorem 4.

1. *Space time coding*: Suppose that the channel information is available at the receiver only. The mutual information in this case is maximized by using independent Gaussian codes on each antenna with equal distribution of power among the antennas[1]. In this case, we have

$$I_{STC}(X;Y|H) = \sum_{i=1}^{\min(m,n)} \log\left(1 + \frac{P}{m}\lambda_i\right) \tag{9}$$

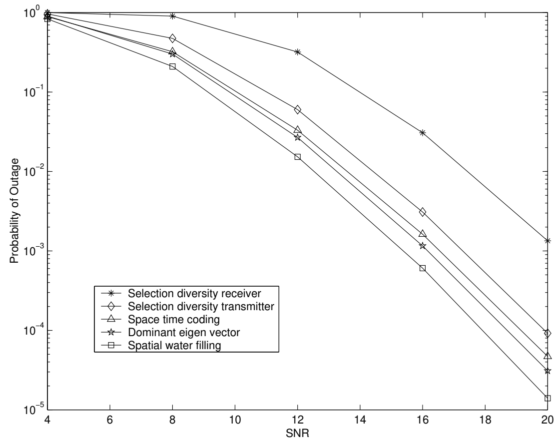
   where λ_i's are the eigenvalues of $HH^\dagger$. It can be shown that

$$\log\left(1 + \frac{P}{m}\|H\|_2^2\right) \leq I_{STC}(X;Y|H) \leq \min(m,n) \log\left(1 + \frac{P}{m}\|H\|_2^2\right) \tag{10}$$

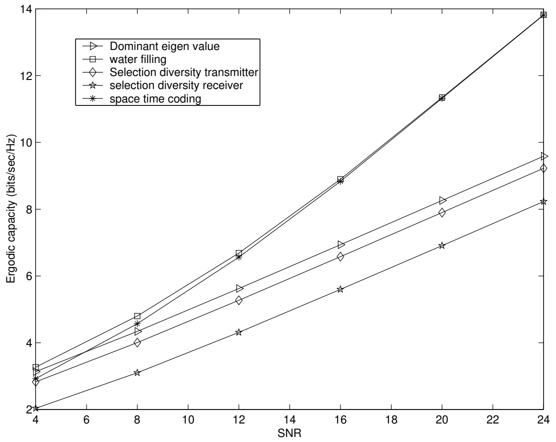
   where $\|.\|_2$ is the induced l_2 norm on $\mathbb{C}^{n\times m}$ and $\|H\|_2^2 = \lambda_{max}$, the maximum eigenvalue of $HH^\dagger$. Hence, the class of signaling schemes $\mathcal{S}$ contains space time coding and by Theorem 4, we have $P_{out}^{STC}(R,P) \sim K_{stc}\left(\frac{1}{P}\right)^{mn}$ for large P, where K_{stc} is a constant dependent on R and independent of P. Note that space time coding does not require feedback resources since the channel state information is not required at the transmitter. But the decoding complexity increases exponentially with the number of transmit antennas.

2. *Spatial water filling*: Suppose that the complete channel state information is available at both the transmitter and the receiver. In this case, the mutual information is maximized by transmitting independent Gaussian codes along the eigenvectors of the channel, with power distribution given by water filling algorithm, proportional to the inverse of the corresponding eigenvalues. The maximum mutual information is given by

$$I_{SWF}(X;Y|H) = \sum_{i=1}^{\min(m,n)} \log(1 + P_i \lambda_i) \tag{11}$$



(a) Outage



(b) Ergodic capacity without time power control

Figure 1: Comparison of the schemes discussed in Section 5 for two transmit and two receive antennas system: (a) Outage probability for R=4 bits/sec/Hz, and (b) ergodic capacity.

where P_i is given by $(\mu - \lambda_i^{-1})^+$ and μ is chosen such that $\sum_{i=1}^{\min(m,n)} P_i = P$. Again, it can be shown that

$$\log\left(1 + P\|H\|_2^2\right) \leq I_{SWF}(X;Y|H) \leq \min(m,n)\log\left(1+P\|H\|_2^2\right). \tag{12}$$

Hence, the class $\mathcal{S}$ contains spatial water filling scheme and we have $P_{out}^{SWF}(R,P) \sim K_{swf}\left(\frac{1}{P}\right)^{mn}$ for large P, where K_{swf} is a constant, independent of P. Spatial water filling may require substantial amount of feedback resources to make the channel state information available at the transmitter. The decoding complexity of this scheme increases linearly with $min(m,n)$.

3. *Beamforming in the direction of dominant eigenvector*: In this case, the transmitter chooses to transmit along the eigenvector of the channel corresponding to the maximum eigenvalue. The mutual information of this scheme is maximized by a Gaussian codebook and is given by

$$I_{BF}(X;Y|H) = \log(1 + P\lambda_{max}) \tag{13}$$

Again, note that $\|H\|_2^2 = \lambda_{max}$ and hence this scheme belongs to $\mathcal{S}$. The decoding complexity of this scheme is independent of the number of transmit antennas though it may still require substantial amount of feedback resources.

4. *Selection diversity at the transmitter*: Suppose that the transmitter chooses to transmit using the antenna with the best channel SNR. In this case, the maximum mutual information is given by

$$I_{SDT}(X;Y|H) = \log\left(1 + P\|h_{cmax}\|_2^2\right) \tag{14}$$

where $h_{cmax} = \arg\max_{h_i}\|h_i\|_2^2$, where h_i is the ith column of H. It can be shown that $\|H\|_{col} = \|h_{cmax}\|_2$, is a norm on $\mathbb{C}^{n\times m}$; hence, $\mathcal{S}$ contains this scheme. The decoding complexity of transmitter selection diversity is again independent of the number of transmit antennas while it requires feedback of $\log_2(m)$ bits, every frame.

5. *Selection diversity at the receiver*: Assume that the transmitter uses space time coding as described in Example 1. Now, suppose that the receiver chooses to use the receive antenna with the best channel SNR. The maximum mutual information in this case is given by

$$I_{SDR}(X;Y|H) = \log\left(1 + \frac{P}{m}\|\bar{h}_{rmax}\|_2^2\right) \tag{15}$$

where $\bar{h}_{rmax} = \arg\max_{\bar{h}_i} \|\bar{h}_i\|^2$, where $\bar{h}_i$ is the ith row of H. Again, it can be shown that $\|H\|_{row} = \|\bar{h}_{rmax}\|_2$ is a norm on $\mathbb{C}^{n\times m}$. Hence, $\mathcal{S}$ contains this scheme and the decay rate of mn of the outage probability as a function of SNR is guaranteed by Theorem 4. Receiver selection diversity does not require any feedback resources in addition to using possibly fewer resources in training the receiver with the channel state information. The decoding complexity of this scheme still grows exponentially in the number of transmit antennas.

The outage probability for the above schemes, with two transmit and two receive antennas transmitting at R=4 bits/sec/Hz is given in Figure 1(a). It can be seen that for reasonably high SNR, the outage probability for all the schemes has a slope of 4 as predicted by Theorem 4. The ergodic capacity of these schemes, as seen from Figure 1(b), exhibits different growth rates with SNR, ergodic capacity of Schemes 1 and 2 (space time coding and spatial water-filling) grows faster than the other three schemes.

6 Conclusion

We have characterized a class of signaling schemes which exploit all the spatial diversity available in the system, in the absence of power control. We observe that the class contains schemes which require very high decoding complexity without any channel state information at the transmitter (space time codes) as well as schemes which have relatively low decoding complexity but require complete channel state information at the transmitter (spatial water filling). In Rayleigh fading, the outage probability of these schemes, which reflects the frame error rates achievable in a practical system, exhibits a similar decay with SNR. Further, the techniques developed in this paper can be used to derive the diversity gain of an arbitrary scheme.

References

[1] E. Telatar, "Capacity of multi-antenna gaussian channels," *AT&T-Bell Labs Internal Tech. Memo*, June 1995.

[2] G.J. Foschini Jr. and M.J. Gans, "On limits of wireless communication in a fading environment when using multiple antennas," in *Wireless Personal Communications*. Kluwer Academic Publishers, March 1998.

[3] V. Tarokh, N. Seshadri, and A. Calderbank, "Space-time codes for high data rate wireless communication:performance criterion and code construction," *IEEE Trans. on Info. Theory*, vol. 44, pp. 744–765, March 1998.

[4] V. Tarokh, H. Jafarkhani, and A.R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Transactions on Information Theory*, vol. 45, no. 5, pp. 1456–1467, July 1999.

[5] T.L. Marzetta and B.M. Hochwald, "Capacity of a mobile multiple-antenna communication link in rayleigh flat fading," *IEEE trasactions on Information Theory*, pp. 139–157, January 1999.

[6] L.H. Ozarow, S. Shamai, and A.D. Wyner, "Information theoretic considerations for cellular mobile radio," *IEEE transactions on Vehicular Technology*, vol. 43, pp. 359–378, May 1994.

[7] S. Bhashyam, A. Sabharwal, and B. Aazhang, "Feedback gain in multiple antenna systems," *submitted to IEEE Trans. Comm.*, 2001.

[8] G. Caire, G. Taricco, and E. Biglieri, "Optimum power control over fading channels," *IEEE Trans. Info. Th.*, vol. 45, no. 5, pp. 1468–1489, July 1999.

[9] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[10] T. Muharemovic and B. Aazhang, "On outage and capacity for orthogonal transmit diversity," *Submitted to IEEE trans. on Information Theory*, 2001, http://www.ece.rice.edu/~tarik/OTD.ps.